Chapter 5. Maximum Value Functions (Exercises)

Exercise 5.1: The Cobb-Douglas Cost Function. Consider a production function

\[ y = A \prod_{j=1}^{n} x_j^{\alpha_j}, \]

where \( y \) is output, \( x_j \)'s are inputs, \( A \) and \( \alpha_j \)'s are positive constants. Let \( w = (w_j) \) be the vector of input prices. Suppose the producer wishes to produce a fixed quantity \( y \) at minimum cost.

**Question 1:** Write out the cost minimization problem and solve for the minimum cost function \( C(w, y) \). Hint: the minimum cost function should be

\[ C(w, y) = \beta \left( \frac{y}{A} \right)^{1/\beta} \prod_{j=1}^{n} \left( \frac{w_j}{\alpha_j} \right)^{\alpha_j/\beta}, \]

where \( \beta = \sum_{j=1}^{n} \alpha_j \).

**Question 2:** If \( \beta < 1 \), calculate the corresponding maximum profit function \( \pi(p, w) \), where \( p \) is the output price. What goes wrong if \( \beta \geq 1 \)?

Exercise 5.2: The CES Expenditure Function. Suppose the direct utility function is

\[ U(x, y) = \left[ \alpha x^\rho + \beta y^\rho \right]^{1/\rho}, \]

where \( x \) and \( y \) are the quantities of the two goods, and \( \alpha > 0, \beta > 0, \rho < 1 \) are given constants. The prices of good \( x \) and \( y \) are \( p \) and \( q \) respectively.

**Question 1:** Show that the expenditure function is of the form

\[ E(p, q, u) = \left[ ap^r + bq^r \right]^{1/r} u, \]

where \( u \) is the target utility level, and \( a, b, \) and \( r \) are constants that can be expressed in terms of \( \alpha, \beta \) and \( \rho \).
**Question 2:** Show that the ratio of the cost-minimizing quantities is

\[ \frac{x}{y} = \frac{a}{b}(\frac{q}{p})^{1-r}. \]

The elasticity of \( \frac{x}{y} \) with respect to \( \frac{q}{p} \):

\[ \frac{d \ln(x/y)}{d \ln(q/p)}. \]

is called the elasticity of substitution in production. Show that in this example, it is constant and equal to \( 1 - r \). What condition must be imposed on \( \rho \) to ensure a non-negative elasticity of substitution, that is, \( r < 1 \)?