Bertrand Competition

The Model

Consider 2 firms, labeled by $i = 1, 2$, selling homogeneous products in a market with unit demand. Suppose that firms' marginal costs are 0, and the firms set prices p_1 and p_2 simultaneously. Consumers purchase from the firm with a lower price p_i, provided that p_i is lower than their valuation.

For simple exposition, we assume consumers are identical and have infinite valuation. Pure strategy equilibrium survives for other types of demand functions. However, for mixed-strategy equilibrium to exist, we need to assume that the consumer demand is relatively inelastic. A more relaxed assumption could be $\lim_{p \to p^M} \Pi(p) = +\infty$, where p^M is the (possibly infinite) monopoly price and $\Pi(p)$ is the profit function for a monopolist. See Baye and Morgan (1999)[1] for details.

The following tie-breaking rule is assumed: if two firms set the same price, each firm gets half of the market.

Therefore, firm i's payoff is as follows:

$$\pi_i = \begin{cases} p_i, & \text{if } p_i < p_j; \\ p_i/2, & \text{if } p_i = p_j; \\ 0, & \text{if } p_i > p_j. \end{cases}$$

Pure Strategy Nash Equilibrium

Claim. The only Pure Strategy Nash Equilibrium is $p_1 = p_2 = 0$.

To see this, consider the following cases.

(i) If $p_i > p_j$. Let $\epsilon = p_i - p_j$, then firm j could benefit by deviating to a price $p'_j = p_j + \sigma$ for some $\sigma \in (0, \epsilon)$. To see this, note that firm j still wins the market since $p_i - p'_j = p_i - p_j - \sigma = \epsilon - \sigma > 0$. And now firm j is able to charge a higher price $p'_j > p_j$.

[1] Baye, M. R., & Morgan, J. (1999). A Folk Theorem for One-shot Bertrand Games. *Economics Letters*, 65(1), 59-65.

(ii) If $p_i = p_j > 0$. Then it would be beneficial for firm i to undercut by a sufficiently small amount ϵ. To see that such ϵ exists, firm i's current profit is $p_i/2$. After deviation, firm i's profit becomes $p_i - \epsilon$. Such undercutting is profitable as long as $p_i - \epsilon - p_i/2 > 0 \implies \epsilon < p_i/2$.

(iii) If $p_i = p_j = 0$. Then, both firms' profits are 0. And any deviation to a higher price is not profitable. To see this, when firm i deviates to $p_i > 0$, it will not gain any market share, and thus profit would stay at 0.

Mixed Strategy Nash Equilibrium

Assume that firms play symmetric strategy. That is, each firm's pricing follow the distribution $p \sim G(p)$. $G(p)$ is the cumulative distribution function.

The fact that firm i mixes means that given firm j's strategy, firm i feels indifferent from choosing any p.

$$\pi_i(p) = p \cdot \underbrace{[1 - G(p)]}_{\substack{i \text{ wins when} \\ j\text{'s price is higher}}} = k, \tag{1}$$

where k is a constant.

We will show below that any $k \in (0, +\infty)$ may be achieved as the profit.

The distribution $G(p)$ is constructed as follows:

(i) Note that p must be unbounded from above. To see this, suppose otherwise, there exists an upper bound p^H. Then $G(p^H) = 1$, and we would have

$$\pi_i(p^H) = p^H \cdot \left[1 - G(p^H)\right] = 0 \neq k.$$

(ii) The lower bound of p, denoted by p^L, could be calculated as follows:

$$\pi_i(p^L) = p^L \cdot \left[1 - \underbrace{G(p^L)}_{G(p^L)=0}\right] = p^L = k.$$

(iii) We can also obtain the distribution $G(p)$ from (1):

$$G(p) = 1 - k/p.$$

Therefore, any profit level $k \in (0, \infty)$ can be achieved per firm in the symmetric mixed strategy Nash equilibrium of a two-firm Bertrand game: The two firms independently mix according to the cumulative distribution function $G(p) = 1 - k/p$ over the interval $[k, \infty)$.