Chapter 1. Static Games of Complete Information

1.A. What is Game Theory?

Game theory is a method of studying strategic situations. Literally, strategic situations are settings where the outcomes that affect you depend on your own actions and the actions of others.

Example 1.A.1.

i. A monopolist is non-strategic. There are no competitors.
ii. Firms under perfect competition are non-strategic. Prices are taken as given, so firms do not need to worry about the actions of the competitors.
iii. Oligopolists are strategic. The actions of the firms affect one another.

Game theory applies in economics, laws, biology, sports, etc. We will discuss some applications in this course.

1.B. Normal-Form Representation of Games

Let us consider the *Prisoners' Dilemma* Game. Two suspects are arrested and charged with a crime. The police lack sufficient evidence to convict the suspects, unless at least one confesses. The police hold the suspects in separate cells and explain the consequences that will follow from the actions they could take.

- If neither confesses then both will be convicted of a minor offense and sentenced to one month in jail.
- If both confess then both will be sentenced to jail for six months.
- If one confesses but the other does not, then the confessor will be released immediately but the other will be sentenced to nine months in jail – six for the crime and a further three for obstructing justice.

We label the two suspects Prisoner 1 and Prisoner 2. They are the *players* of the game. The *strategies* each of the prisoners could take are "Not confess" or Cooperate, denoted by C, and "Confess" or Defect, denoted by D. The information on the outcomes could be concisely recorded in the following tables: Figure 1.1 and 1.2. Figure 1.1 is the outcome matrix for Prisoner 1.

		Prisoner 2	
		Not confess: Cooperate (C)	Confess: Defect (D)
Prisoner 1	Cooperate (C)	1 month in prison	9 months in prison
	Defect (D)	released	6 months in prison

Figure 1.1: Prisoner 1's Outcome

The first cell "1 month in prison" denotes the outcome for Prisoner 1 when Prisoner 1 chooses C (meaning not confess, or cooperate) and Prisoner 2 also chooses C. The rest of the cells could be similarly written out. Following the same procedure, we could write out the outcome matrix for Prisoner 2, shown in Figure 1.2 below.

		Prisoner 2	
		Cooperate (C)	Defect (D)
Prisoner 1	Cooperate (C)	1 month in prison	released
	Defect (D)	9 months in prison	6 months in prison

Figure 1.2: Prisoner 2's Outcome

Rather than drawing two tables, we could super-impose the second table on top of the first table, forming the outcome matrix. See Figure 1.3.

		Prisoner 2	
		Cooperate (C)	Defect (D)
Prisoner 1	Cooperate (C)	(1 month, 1 month)	(9 months, released)
	Defect (D)	(released, 9 months)	(6 months, 6 months)

Figure 1.3: The Outcome Matrix

In each cell, the first outcome belongs to the row player, i.e., Prisoner 1 in our example, and the second outcome belongs to the column player, i.e., Prisoner 2 in our example.

To analyze the game, we are still missing the *payoffs*. That is, we still need to know what the players care about. In this prisoners' dilemma game, we assume that the prisoners care about their own jail time, and they get utility -1 for 1 month in prison. Based on the outcome matrix, we could write down the payoff matrix for the prisoners' dilemma game. See Figure 1.4.

		Prisoner 2	
		Cooperate (C)	Defect (D)
Prisoner 1	Cooperate (C)	$(-1, -1)$	$(-9, 0)$
	Defect (D)	$(0, -9)$	$(-6, -6)$

Figure 1.4: The Payoff Matrix

Figure 1.4 contains all the information we need to analyze the prisoners' dilemma game. It is the *normal-form representation* of the game. Formally, the *normal-form representation* of a game specifies:

1. the players in the game,
2. the strategies available to each player,
3. the payoffs received by each player for each combination of strategies that could be chosen by the players.

Table 1.1 summarizes the notations of these 3 ingredients of a game.

	Notations	in Prisoners' Dilemma Game
Players	Player i for $i = 1, .., n$	Prisoner 1 and 2
Strategies	S_i: i's strategy space (set of possible strategies of Player i)	$\{C, D\}$ for $i = 1, 2$
	s_i: a particular strategy for Player i	C or D for $i = 1, 2$
	$s = (s_1, ...s_i, ..., s_N)$: a strategy profile (a particular play of the game)	e.g. (C, C)
Payoffs	$u_i(s) = u_i(s_1, ...s_i, ..., s_N)$	e.g. $u_1(C, C) = -1$

Table 1.1: The Ingredients of a Game

The definition of the *normal-form representation* of a game is given by Definition 1.B.1.

Definition 1.B.1. The **normal-form representation** of an n-player game specifies the players' **strategy space** $S_1, ..., S_n$ and their **payoff functions** $u_1, ..., u_n$. We denote this game by $G = \{S_1, ..., S_n; u_1, ..., u_n\}$.

Remark 1.1 (Timing vs. Information)**.** To analyze a game, information (what does Player i know) is more important than timing (when do the players move). For example, in the prisoners' dilemma game, the prisoners do not need to move *simultaneously*, it suffices that each prisoner choose his/her action without knowledge of the other's choices. That is, if the players do not know what the other players choose even though the players move sequentially, it would be as if the players choose their strategies simultaneously. This point will be addressed later in the course.

1.C. Iterated Elimination of Strictly Dominated Strategies

We are now ready to analyze the prisoners' dilemma game. To solve the prisoners' dilemma game, we will use the idea that **a rational player will not play a strictly dominated strategy.**

Definition 1.C.1. In the normal-form game $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, let s'_i and s''_i be feasible strategies for Player i (i.e., $s'_i, s''_i \in S_i$). Strategy s'_i is **strictly dominated** by strategy s''_i if for each feasible combination of the other players' strategies, i's payoff from playing s'_i is strictly less than i's payoff from playing s''_i:

$$u_i(s_1, ...s_{i-1}, s'_i, s_{i+1}, ..., s_n) < u_i(s_1, ...s_{i-1}, s''_i, s_{i+1}, ..., s_n) \tag{DS}$$

for each $(s_1, ...s_{i-1}, s_{i+1}, ..., s_n)$ that can be constructed from the other players's strategy spaces $S_1, ..., S_{i-1}, S_{i+1}, ..., S_n$.

Definition 1.C.1 tells us that strategy s'_i is **strictly dominated** by strategy s''_i if the payoff from strategy s''_i is strictly higher than that from strategy s'_i **regardless of the other players' choices**. We also say strategy s''_i **strictly dominates** strategy s'_i.

1.C.1. The Prisoners' Dilemma Game

We apply the idea that "a rational player will not play a strictly dominated strategy" to the prisoners' dilemma game in Figure 1.4. For Prisoner 1,

- If Prisoner 2 chooses C, then Prisoner 1 choosing C yields -1 and D yields 0;
- If Prisoner 2 chooses D, then Prisoner 1 choosing C yields -9 and D yields -6.

In either case, choosing D is strictly better for Prisoner 1. That is, Prisoner 1's strategy C is *strictly dominated* by the strategy D, or Prisoner 1's strategy D *strictly dominates* the strategy C. Therefore, Prisoner 1 should choose D. Following the same logic, Prisoner 2 should also choose D. Thus, (D, D) will be the solution of the game.

Remark 1.2. The process is called *elimination of strictly dominated strategies.*

Remark 1.3. The only individually rational solution (D, D) is *Pareto inefficient.* That is, both prisoners would obtain higher payoffs if they choose (C, C).

The prisoners' dilemma game has many applications, including

- arms race (D: high level of arms, C: low level of arms);
- price wars (D: undercut price, C: set high price);
- free-rider problem in the provision of public goods[1] (D: do not provide public goods, C: provide public goods);
- joint project (D: shirk, C: cooperate).

Question 1.1. Can you think of any ways to make the good outcome (C, C) happen?

Question 1.2. Notice that direct communication between the players would not work. Why?

1.C.2. Other Examples

We apply the idea that a rational player will not play a strictly dominated strategy on other games.

Example 1.C.1. Consider the following game:

		Player 2	
		Left	Right
Player 1	Up	$(1, 0)$	$(1, 2)$
	Down	$(0, 3)$	$(0, 1)$

Figure 1.5: Example 1.C.1

In this example,

- Players: Player 1 and Player 2;
- Strategy spaces: $S_1 = \{\text{Up, Down}\}$, $S_2 = \{\text{Left, Right}\}$.
- Payoffs: $u_1(\text{Up, Left}) = 1$, $u_2(\text{Up, Left}) = 0$ for example.

[1] A public good is a good that is non-excludable (it is not possible to exclude someone from enjoying the benefits) and non-rivalrous (when one person uses the good, another can also use it). Examples of public goods include clean air, streetlights, invention, herd immunity, etc.

For Player 1, “Up” strictly dominates “Down”. So, a rational Player 1 would not choose “Down”. For Player 2,

- “Right” is better than “Left” if Player 1 plays “Up”;
- “Left” is better than “Right” if Player 1 plays “Down”.

Question 1.3. What would Player 2 do?

If Player 2 knows that Player 1 is rational, then Player 2 could eliminate “Down” from Player 1’s strategy space. The game becomes

		Player 2	
		Left	Right
Player 1	Up	$(1, 0)$	$(1, 2)$

Figure 1.6: “Down” Eliminated

Then, a rational Player 2 would choose “Right”.

Summing up, the solution of the game is (Up, Right).

Remark 1.4. Even though neither “Left” or “Right” is strictly dominated for Player 2, by figuring out what Player 1 would do, Player 2 would choose “Right” (as long as Player 2 is rational and Player 2 knows that Player 1 is rational).

Example 1.C.2. Consider the following game:

		Player 2		
		Left	Middle	Right
Player 1	Up	$(1, 0)$	$(0, 1)$	$(1, 2)$
	Down	$(0, 3)$	$(2, 0)$	$(0, 1)$

Figure 1.7: Example 1.C.2

For Player 1, neither “Up” or “Down” is strictly dominated:

- “Up” is better than “Down” if Player 2 plays “Left”;
- “Down” is better than “Up” if Player 2 plays “Middle”.

For Player 2, “Middle” is strictly dominated by “Right”: “Middle” is better than “Right” no matter whether Player 1 plays “Up” or “Down”. So, a rational Player 2 would not play

"Middle". Thus, if Player 1 knows that Player 2 is rational, then Player 1 could eliminate "Middle" from Player 2's strategy space. The game becomes the one in Example 1.C.1. Then, if Player 1 is rational (and Player 1 knows that Player 2 is rational, so that the game in Figure 1.5 applies) then Player 1 will not play "Down".
Thus, if Player 2 knows that Player 1 is rational, and Player 2 knows that Player 1 knows that Player 2 is rational (so that Player 2 knows that Figure 1.5 applies), then Player 2 can eliminate "Down" from Player 1's strategy space, leaving the game in Figure 1.6. Then a rational Player 2 would choose "Right". Therefore, the solution of the game is (Up, Right).

Remark 1.5. The process is called *iterated elimination of strictly dominated strategies.*

Remark 1.6. Compared to the elimination of strictly dominated strategies, as we did in the prisoners' dilemma game, the *iterated* elimination of strictly dominated strategies has stronger predictive power. In our examples, only eliminating strictly dominated strategies does not tell us what Player 2 would do in Example 1.C.1. The same is true for both Player 1 and Player 2 in Example 1.C.2.

1.C.3. Application of Iterated Elimination of Strictly Dominated Strategeis: Voting

Game Setup. Suppose that two candidates are choosing their political positions for an election. There are 10 positions to choose from, namely, 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10. The voters are uniformly distributed. That is, there are 10% of voters at each position. The voters will vote for the closest candidate, i.e., the candidate whose position is closest to their own. If there is a tie, the voters of that position split evenly. The candidates' objective is to maximize the share of votes.
To test whether you have understood the game, answer the following two questions.

Question 1.4. Who are the players? What are the strategy spaces and payoffs?

Question 1.5. Suppose that one of the candidate is at position 2 and the other at position 6. What are their shares of votes? (The answers are 35% and 65%.)

Analysis. We solve the game using *Iterated Elimination of Strictly Dominated Strategies.*

Question 1.6. Does position 2 dominate position 1?

We need to work out the share of votes a candidate would get if he/she chooses position 1 or position 2, against **all** different positions the other candidate could choose. Let the candidate under concern be Candidate 1, and denote $u_1(a_1, a_2)$ the share of votes candidate 1 gets when he/she chooses a_1 and Candidate 2 chooses a_2.

Then, if Candidate 2 chooses position 1, Candidate 1 gets $u_1(1,1) = 50\%$ when choosing position 1 and $u_1(2,1) = 90\%$ when choosing position 2. Since $u_1(1,1) < u_1(2,1)$, Candidate 1 is better-off choosing position 1.

Similarly, if Candidate 2 chooses position 2, $u_1(1,2) = 10\% < u_1(2,2) = 50\%$.

For the other positions that Candidate 2 could choose (position 3 and above), Candidate 1 is always better-off choosing position 2:

$$u_1(1,3) = 15\% < u_1(2,3) = 20\%;$$
$$u_1(1,4) = 20\% < u_1(2,4) = 25\%;$$
$$\vdots$$

Therefore, position 2 indeed strictly dominates position 1. And following the same argument, position 9 strictly dominates position 10.

Question 1.7. Does position 3 dominate position 2?

No. If Player 2 chooses position 1, then position 2 does better than position 3:

$$u_1(2,1) = 90\% > u_1(3,1) = 85\%.$$

Question 1.8. What if positions 1 and 10 are deleted[2] (since they are strictly dominated by position 2 and position 9 respectively)? Does position 3 dominate position 2 then?

Yes. Still, we need to check the share of votes Candidate 1 would get if he/she chooses position 2 or position 3, against **all** different positions except position 1 and 10:

$$u_1(2,2) = 50\% < u_1(3,2) = 80\%;$$
$$u_1(2,3) = 20\% < u_1(3,3) = 50\%;$$
$$u_1(2,4) = 25\% < u_1(3,4) = 30\%;$$
$$\vdots$$

[2]Note that by deletion, we mean that we delete the candidates' strategies "position 1" and "position 10". The voters at positions 1 and 10 are still there.

Therefore, position 3 strictly dominates 2 once we realize that positions 1 and 10 are dominated. Following the same argument, position 8 strictly dominates 9. We delete positions 2 and 9 since they are strictly dominated (after we delete positions 1 and 10). Then we could iterate once more and delete positions 3 and 8. And after that, we could delete positions 4 and 7. In the end, we are left with positions 5 and 6. So the prediction from our analysis is that the candidates would both choose the center positions.

Remark 1.7. In political science, it is called the *Median Voter Theorem*.

Remark 1.8. This idea was introduced by Downs (1957) in political science. Hotelling (1929) raised a similar idea in economics on product positioning.

1.C.4. Drawbacks of Iterated Elimination of Strictly Dominated Strategies

Applying the iterated elimination of strictly dominated strategies, we could make predictions about the outcomes of certain games. However, the process has two drawbacks.

Common Knowledge of Rationality. Compared to Elimination of Strictly Dominated Strategies, *Iterated* Elimination requires further assumptions on what the players know about each other's rationality. As a concrete example, see the descriptions in Section 1.C.2. To iterate an arbitrary number of rounds, we need to assume not only that all the players are rational, but also that all the players know that all the players are rational, and that all the players know that all the players know that all the players are rational, and so on, *ad infinitum*. This is called *common knowledge* of rationality.

Note that there is a difference between *common knowledge* and *mutual knowledge*. An event is *mutual knowledge* if everyone knows it.

Example 1.C.3 (The Hat Puzzle)**.** Two individuals wear hats of two possible colors: black or white. Each individual observes the color of the other individual's hat but not the color of his own hat. Suppose that both of them wear a white hat.

- Situation 1: An outsider says "I will count slowly. Raise your hand if you know the color of your hat". No one raises their hands and the counts go on forever.
- Situation 2: Before counting, the outsider mentions "At least one of you wears a white hat". Both players raise their hands in round 2.

In this example, "At least one of you wears a white hat" is mutual knowledge in situation 1 and becomes common knowledge in situation 2.

Prediction Power. Applying Iterated Elimination of Strictly Dominated Strategies, we are only able to solve a limited number of games. For example, it produces no prediction about the following game.

Example 1.C.4. The game in Figure 1.12 has no strictly dominated strategies.

		Player 2	
		Left	Right
	Up	$(5,1)$	$(0,2)$
Player 1	Middle	$(1,3)$	$(4,1)$
	Down	$(4,2)$	$(2,3)$

Figure 1.8: Example of Games with No Strictly Dominated Strategies

To show that there exists no strictly dominated strategies, we need to check the strategies of each player one by one.

For Player 1,

- "Up" is not dominated by "Middle" since it does better against "Left";
- "Up" is not dominated by "Down" since it does better against "Left";
- "Middle" is not dominated by "Up" since it does better against "Right";
- "Middle" is not dominated by "Down" since it does better against "Right";
- "Down" is not dominated by "Up" since it does better against "Right";
- "Down" is not dominated by "Middle" since it does better against "Left".

For Player 2,

- "Left" is not dominated by "Right" since it does better against "Middle";
- "Right" is not dominated by "Left" since it does better against "Up".

Question 1.9. Would you choose "Up", "Middle" or "Down" if you were Player 1?

Since no strategy of either player is strictly dominated, to answer this question, we could not rely on *Iterated Elimination of Strictly Dominated Strategies*. We will look at this problem again in Section 1.D.1.

1.C.5. Weakly Dominated Strategies

Similar to strictly dominated strategies (Definition 1.C.1), we define *weakly dominated strategies* as follows.

Definition 1.C.2. Strategy s_i' is **weakly dominated** by strategy s_i'' if

$$u_i(s_i', s_{-i}) \leq u_i(s_i'', s_{-i}) \text{ for all } s_{-i};$$
$$u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i}) \text{ for some } s_{-i}.$$

where $s_{-i} = (s_1, ...s_{i-1}, s_{i+1}, ..., s_n)$ denotes a strategy profile of all other players except i.

Definition 1.C.2 tells us that strategy s_i' is **weakly dominated** by strategy s_i'' if the payoff from strategy s_i'' is weakly higher than that from strategy s_i' for all of the other players' choices and strictly higher for some of the other players' choices. We also say strategy s_i'' **weakly dominates** strategy s_i'.

Question 1.10. Could we have "iterated elimination of weakly dominated strategies"?

The problem with iterative elimination of weakly dominated strategy is that the prediction may depend on the order in which actions are eliminated.

Example 1.C.5. Consider the following game:

		Player 2		
		Left	Middle	Right
Player 1	Up	$(0, 1)$	$(1, 0)$	$(0, 0)$
	Down	$(0, 0)$	$(0, 0)$	$(1, 0)$

Figure 1.9: Example 1.C.5

For Player 2, both "Middle" and "Right" are weakly dominated by "Left".

- If only "Right" is removed,

	Left	Middle
Up	$(0, 1)$	$(1, 0)$
Down	$(0, 0)$	$(0, 0)$

$\xrightarrow[\text{dominated by U}]{\text{D weakly}}$

	Left	Middle
Up	$(0, 1)$	$(1, 0)$

$\xrightarrow[\text{dominated by L}]{\text{M strictly}}$

	Left
Up	$(0, 1)$

Figure 1.10: Eliminate "Right" First

- If only "Middle" is removed,

	Left	Right
Up	$(0,1)$	$(0,0)$
Down	$(0,0)$	$(1,0)$

$\xrightarrow[\text{dominated by D}]{\text{U weakly}}$

	Left	Right
Down	$(0,0)$	$(1,0)$

Figure 1.11: Eliminate "Left" First

- If both "Right" and "Middle" are removed, the prediction is

	Left
Up	$(0,1)$
Down	$(0,0)$

Before moving on to an economic application, let us play the following guessing game.

Example 1.C.6. Everyone in the class pick an integer from $[1,100]$. The winner is the person whose number is closest to two-thirds times the average in the class.

In this game, iterated elimination of weakly dominated strategies results in a unique solution.

1.C.6. Application of Weakly Dominated Strategies: Second-Price Auction

Game Setup. There is one indivisible good for sale. The valuations of N potential buyers are independently drawn from a uniform distribution with support $[0,1]$.[3] Denote Buyer i's valuation by v_i. The auction rule is as follows:

- Buyers bid simultaneously and each submits a bid $b_i \in [0,+\infty)$.
- The bidder with the highest bid wins the auction and pays the second highest bid.
- If k buyers submit the same highest bid, then each of the k buyers has $1/k$ chance of winning the good. The payment is the highest bid (since there is a tie).

Analysis. Buyer i 's payoff when submitting the bid b_i is

$$u_i = \begin{cases} 0 & \text{if } b_i < \max_{j\neq i} b_j \\ \frac{v_i - \max_{j\neq i} b_j}{k} & \text{if } b_i = \max_{j\neq i} b_j \\ v_i - \max_{j\neq i} b_j & \text{if } b_i > \max_{j\neq i} b_j \end{cases}$$

where k is the number of buyers bidding b_i when $b_i = \max_{j\neq i} b_j$.

[3] Sealed-bid auctions actually belong to games of incomplete information. See Chapter 3 for details. Nevertheless, the second-price auction described here is dominant strategy solvable.

Claim. $b_i \neq v_i$ is weakly dominated by $b_i = v_i$.

The table below summarizes buyer i's payoff when bidding $b_i'' < v_i$, $b_i = v_i$ and $b_i' > v_i$.

	bidding $b_i'' < v_i$	bidding $b_i = v_i$	bidding $b_i' > v_i$
$\max_{j\neq i} b_j > b_i'$	0	0	0
$\max_{j\neq i} b_j = b_i'$	0	0	$-(\max_{j\neq i} b_j - v_i)/k$
$\max_{j\neq i} b_j \in (v_i, b_i')$	0	0	$-(\max_{j\neq i} b_j - v_i)$
$\max_{j\neq i} b_j = v_i'$	0	0	0
$\max_{j\neq i} b_j \in (b_i'', v_i)$	0	$v_i - \max_{j\neq i} b_j$	$v_i - \max_{j\neq i} b_j$
$\max_{j\neq i} b_j = b_i''$	$(v_i - \max_{j\neq i} b_j)/k$	$v_i - \max_{j\neq i} b_j$	$v_i - \max_{j\neq i} b_j$
$\max_{j\neq i} b_j < b_i''$	$v_i - \max_{j\neq i} b_j$	$v_i - \max_{j\neq i} b_j$	$v_i - \max_{j\neq i} b_j$

It is not hard to see that $b_i = v_i$ weakly dominates both $b_i'' < v_i$ and $b_i' > v_i$. Therefore, the optimal strategy in a second-price auction is to bid one's own valuation.

1.D. Best Responses

1.D.1. Example 1.C.4 Revisited

Consider again the following game that we have analyzed in Example 1.C.4:

		Player 2	
		Left	Right
	Up	$(5, 1)$	$(0, 2)$
Player 1	Middle	$(1, 3)$	$(4, 1)$
	Down	$(4, 2)$	$(2, 3)$

Figure 1.12: Example 1.C.4

We already know that none of the strategies are strictly dominated.

Question 1.11. Suppose that you are Player 1. Could you justify the behavior of choosing "Up"? How about "Middle" and "Down"?

To answer this question, we need to consider your belief on Player 2's strategy. Let your belief on the probability that Player 2 would choose "Right" be $\Pr(\text{Right}) = p_r$. Then you belief on the probability that Player 2 would choose "Left" is $1 - p_r$.

"Up" dominates "Middle" when

$$5 \cdot (1 - p_r) + 0 \cdot p_r \geq 1 \cdot (1 - p_r) + 4 \cdot p_r \implies p_r \leq \frac{1}{2};$$

"Up" dominates "Down" when

$$5 \cdot (1 - p_r) + 0 \cdot p_r \geq 4 \cdot (1 - p_r) + 2 \cdot p_r \implies p_r \leq \frac{1}{3}.$$

Therefore, "Up" dominates both "Middle" and "Down" when $p_r \leq \frac{1}{3}$. And thus, the belief $p_r \leq \frac{1}{3}$ justifies the choice of "Up". Formally, "Up" is called a *Best Response (BR)* to the belief $p_r \leq \frac{1}{3}$.

Definition 1.D.1. Player i's strategy $\hat{s}_i$ is a Best Response (BR) to the belief p about the other players' choices if

$$\mathbb{E}u_i(\hat{s}_i, p) \geq \mathbb{E}u_i(s_i', p) \text{ for all } s_i' \in S_i,$$

or $\hat{s}_i$ solves

$$\max_{s_i} \mathbb{E}u_i(s_i, p).$$

Remark 1.9. Definition 1.D.1 does not limit to two-player games. Besides, each player could have any number of strategies.

Question 1.12. Under which belief is "Down" a Best Response? How about "Middle"?

Answer: "Down" is a BR when $p_r \in \left[\frac{1}{3}, \frac{2}{5}\right]$. "Middle" is a BR when $p_r \geq \frac{2}{5}$.
The dominance relationships could be more clearly shown in a figure. First, we calculate Player 1's expected payoffs from each strategy.

$$\mathbb{E}u_1(\text{Up}, p_r) = 5 \cdot (1 - p_r) + 0 \cdot p_r = 5 - 5p_r$$

$$\mathbb{E}u_1(\text{Middle}, p_r) = 1 \cdot (1 - p_r) + 4 \cdot p_r = 1 + 3p_r$$

$$\mathbb{E}u_1(\text{Down}, p_r) = 4 \cdot (1 - p_r) + 2 \cdot p_r = 4 - 2p_r$$

Then we draw the expected payoffs in a two-dimensional figure, as shown in Figure 1.13. The horizontal axis represents $p_r \in [0, 1]$ and the vertical axis is the value. The expected payoffs turn out to be three lines. At each p_r, we could trace the expected payoffs from the three strategies respectively. The strategy corresponding to the highest payoff is the Best Response.

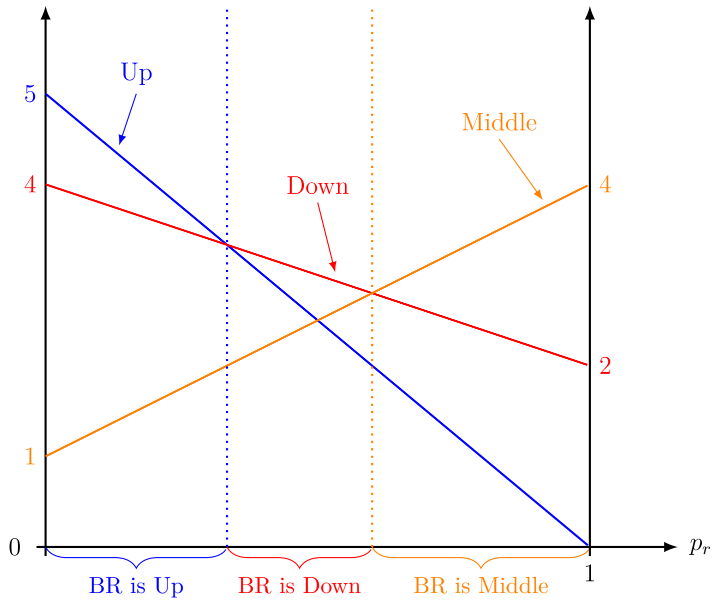

Figure 1.13: Best Responses

1.D.2. Penalty Kick Game

Example 1.D.1 (Penalty Kick Game)**.** Consider the following penalty kick game.

		Goalkeeper	
		Left (L)	Right (R)
	left (l)	$(4, -4)$	$(9, -9)$
Shooter	middle (m)	$(6, -6)$	$(6, -6)$
	right (r)	$(9, -9)$	$(4, -4)$

Figure 1.14: Penalty Kick Game

Similar to our analysis in the previous example, we calculate the expected payoffs:

$$\mathbb{E}u_S(l, p_R) = 4 \cdot (1 - p_R) + 9 \cdot p_R = 4 + 5p_R$$

$$\mathbb{E}u_S(m, p_R) = 6 \cdot (1 - p_R) + 6 \cdot p_R = 6$$

$$\mathbb{E}u_S(r, p_R) = 9 \cdot (1 - p_R) + 4 \cdot p_R = 9 - 5p_R$$

We could then draw the expected payoffs in the figure, as shown in Figure 1.15.

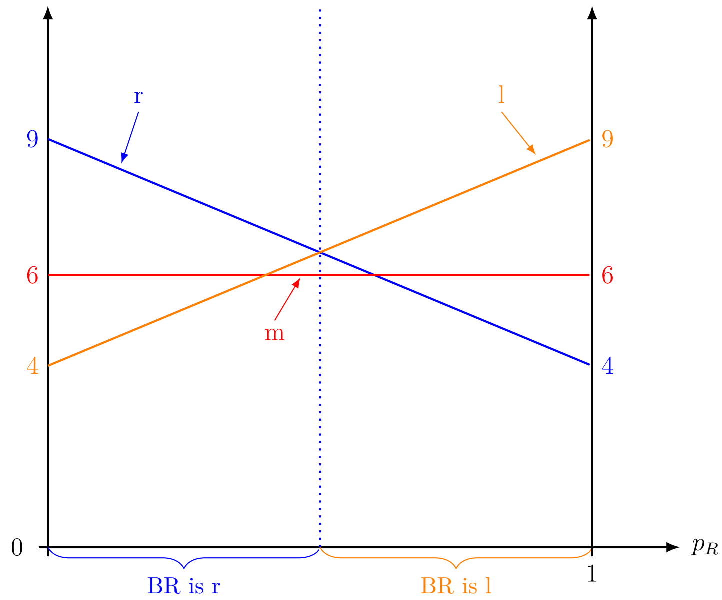

Figure 1.15: Penalty Kick Game

It is clear from Figure 1.15 that m is not a best response to any belief. Therefore, the shooter should not kick to the middle.

Remark 1.10. One should not choose a strategy that is never a Best Response (BR) to any belief.

1.D.3. Partnership Game

We will learn to apply the idea that "players do not choose a strategy that is never a BR" in the following partnership game.

Game Setting. 2 agents form a partnership. For the partnership to work, each agent $i = 1, 2$ needs to put in effort $s_i \in S_i = [0, 4]$. The cost of effort is $-s_i^2$. The total profit from the partnership is

$$4(s_1 + s_2 + bs_1 s_2),$$

where $b \in [0, \frac{1}{4}]$ indicates the synergy between the 2 agents. The agents share the profit equally, each obtaining 50% of the profit.

Analysis. The payoffs for the 2 agents are

$$u_1(s_1, s_2) = \frac{1}{2}[4(s_1 + s_2 + bs_1 s_2)] - s_1^2;$$
$$u_2(s_1, s_2) = \frac{1}{2}[4(s_1 + s_2 + bs_1 s_2)] - s_2^2.$$

Therefore, the best responses $\hat{s}_1$ and $\hat{s}_2$ solves

$$\max_{s_1} 2(s_1 + s_2 + bs_1 s_2) - s_1^2;$$
$$\max_{s_2} 2(s_1 + s_2 + bs_1 s_2) - s_2^2.$$

First Order Condition (FOC) gives

$$\hat{s}_1 = 1 + bs_2 = \text{BR}_1(s_2);$$
$$\hat{s}_2 = 1 + bs_1 = \text{BR}_2(s_1).$$

We graphically illustrate the case with $b = \frac{1}{4}$. See Figure 1.16.

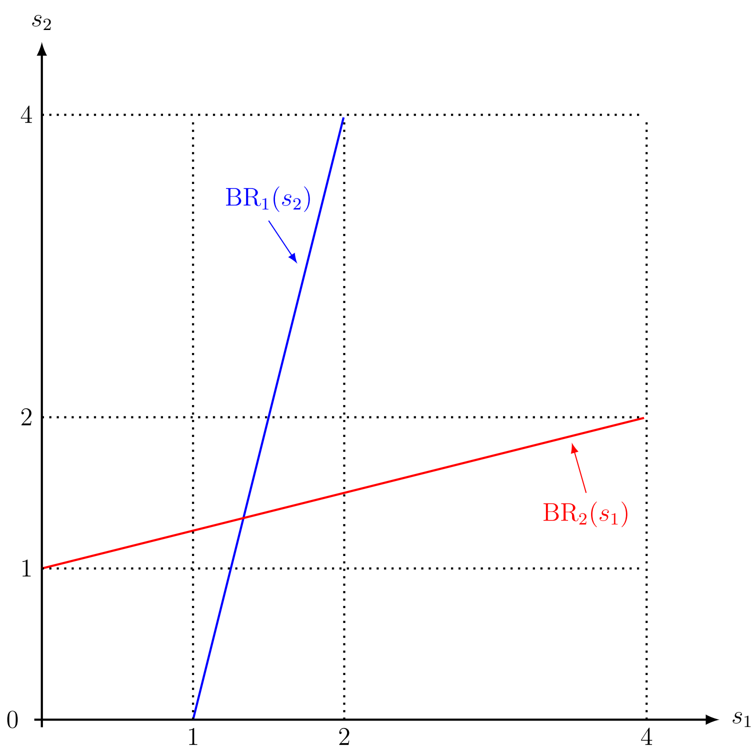

Figure 1.16: Partnership Game

For each agent, $s_i < 1$ and $s_i > 2$ are never best responses. So, these strategies should not be played. Let us delete these strategies. After deleting those strategies that are never best responses, we are left with the strategies in the small black box in Figure 1.17.

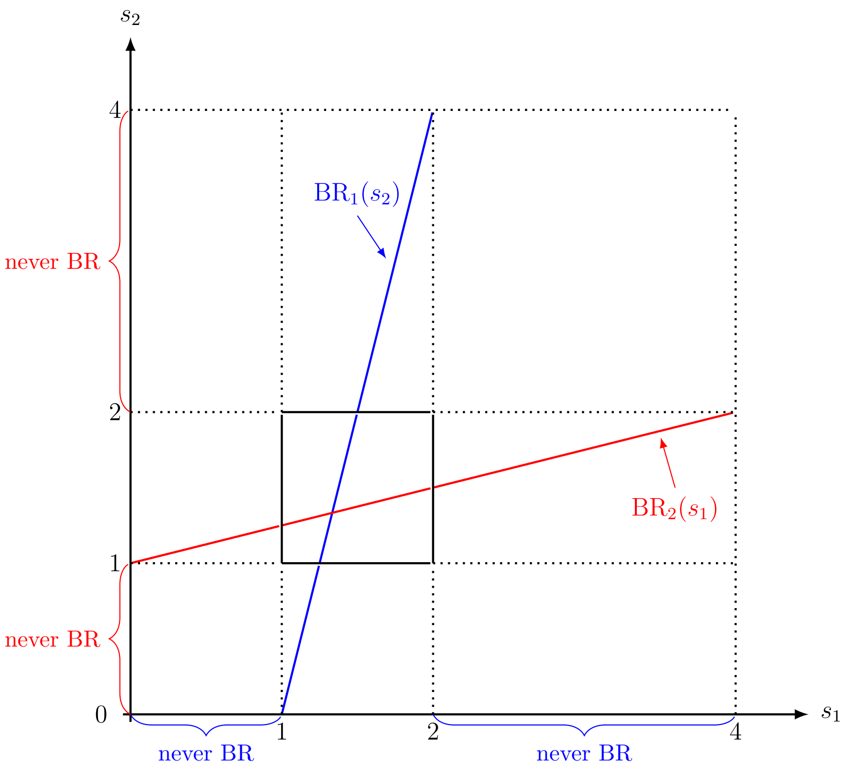

Figure 1.17: Partnership Game - First Round Deletion

Now look closely at this small box. Some of the strategies that survives the deletion were best responses to some strategies, but the strategies they were best responses to have been deleted. For example, $s_1 = 1.8$ was a best response to $s_2 = 3.2$. However, $s_2 = 3.2$ was deleted as it is larger than 2. We further delete those strategies that were never best responses to the opponent's strategies after the first round of deletion. This procedure is similar to the *Iterated Elimination of Strictly Dominated Strategies* we learned before.

After the second round of deletion, we are left with a smaller black box shown in Figure 1.18. Again, we could delete all strategies that were never best responses to the opponent's strategies after the second round of deletion. Then, we would get an even smaller box after the third round of deletion. We could apply the same procedure again and again after each round of deletion. Eventually, we will end up with the intersection:

$$\begin{cases} s_1^* = 1 + b s_2^*; \\ s_2^* = 1 + b s_1^*. \end{cases} \Longrightarrow (s_1^*, s_2^*) = (\frac{1}{1-b}, \frac{1}{1-b}).$$

Remark 1.11. At the intersection, both players play best responses to each other.

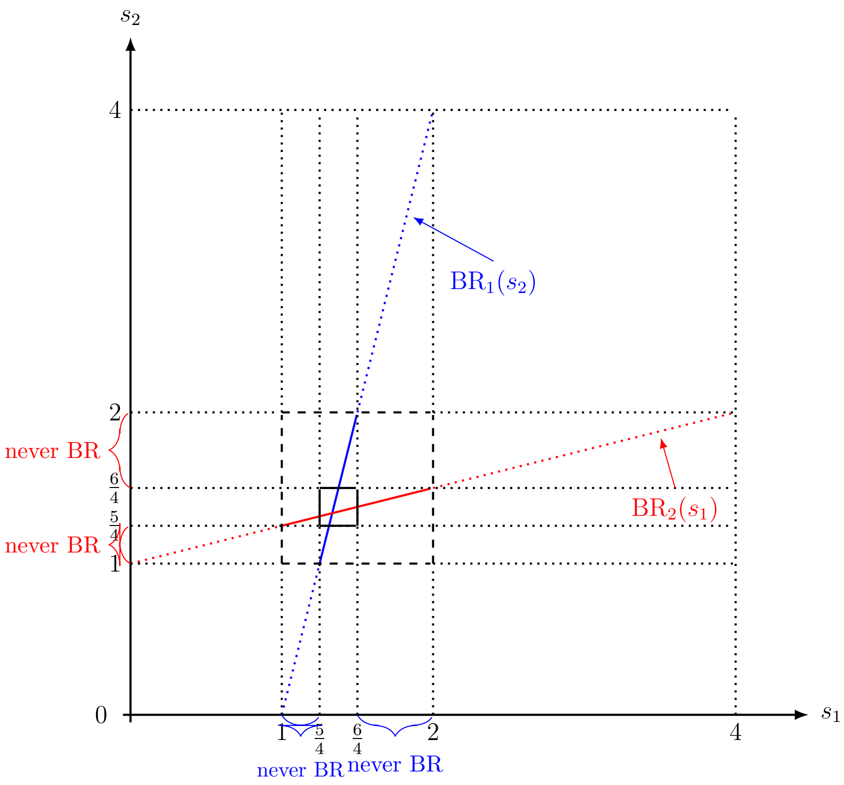

Figure 1.18: Partnership Game - Second Round Deletion

Social Optimum. Suppose that there is a social planner who could decide how much each agent should work in the partnership. The social planner's objective is to maximize the total profit net of the costs. That is, the social planner solves

$$\max_{s_1,s_2} U(s_1, s_2) = \max_{s_1,s_2} 4(s_1 + s_2 + bs_1s_2) - s_1^2 - s_2^2. \tag{1.D.1}$$

Let the optimal effort be s_1^{**} and s_2^{**}.

Question 1.13. How does s_1^* and s_2^* compare to s_1^{**} and s_2^{**}? Or put it differently, do the agents work too much or too little in the partnership compared to the social optimum?

FOCs to the social planner's problem (1.D.1) are

$$\begin{cases} s_1^{**} = 2 + 2bs_2^{**} \\ s_2^{**} = 2 + 2bs_1^{**} \end{cases} \implies (s_1^{**}, s_2^{**}) = (\frac{2}{1-2b}, \frac{2}{1-2b})$$

It is not hard to check that $s_1^{**} > s_1^*$ and $s_2^{**} > s_2^*$. So, the agents work too little compared to the social optimum. The **intuition** is that in the partnership, at the margin, each agent bear the full cost for the extra effort he/she puts in, but the benefit is shared with the other agent. This leads the agents to put in too little effort.

Remark 1.12. There are three things that we usually do when we face a problem (and also when we write an applied paper): do the mathematical calculation, draw figures, and understand the intuition.

1.E. Nash Equilibrium

We solve the partnership game in Section 1.D.3 by iterated elimination of never best responses. In this particular game, we get convergence. At the intersection, both players play best responses to each other. The property of mutual best responses gives rise to the concept of a *Nash Equilibrium*, which is formally defined in Definition 1.E.1.

Definition 1.E.1. In the n-player normal form game $G = \{S_1, ..., S_n; u_1, .., u_n\}$, the strategies $(s_1^*, ..., s_n^*)$ are a (Pure Strategy) **Nash Equilibrium** if, for each player i, s_i^* is (at least tied for) Player i's *best response* to the strategies specified for the $n-1$ other players, $(s_1^*, ..., s_{n-1}^*, s_{n+1}^*, ..., s_n^*)$:

$$u(s_1^*, ..., s_{i-1}^*, s_i^*, s_{i+1}^*, ..., s_n^*) \geq u(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*) \tag{NE}$$

for every feasible strategy $s_i \in S_i$; that is, s_i^* solves

$$\max_{s_i \in S_i} u(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*).$$

Remark 1.13 (No Regret)**.** At a Nash equilibrium, no player can do strictly better by deviating, holding everyone else's actions fixed.

Remark 1.14. A Nash equilibrium can be thought of as self-fulfilling beliefs: Player i would play his/her Nash strategy if he/she believes that the other players play their Nash strategies.

Example 1.E.1. Find the Nash equilibrium in the following game.

		Player 2		
		Left	Center	Right
Player 1	Up	$(0, \underline{4})$	$(\underline{4}, 2)$	$(5, 3)$
	Middle	$(\underline{4}, 0)$	$(0, \underline{4})$	$(5, 3)$
	Down	$(3, 5)$	$(3, 5)$	$(\underline{6}, \underline{6})$

Figure 1.19: Nash Equilibrium

In Figure 1.19, we underline the payoffs of Player 1's (Player 2's) best responses to each of Player 2's (Player 1's) strategies. The Nash equilibrium is where the best responses coincide, i.e., (Down, Right).

Remark 1.15. This game could not be solved using *Iterated Elimination of Strictly Dominated Strategies* or using *Iterated Elimination of Never Best Responses*.

Remark 1.16. For example, a rational Player 1 could choose "Middle" because Player 1 thinks that Player 2 would choose "Left". And Player 1 thinks that Player 2 would choose "Left" because Player 2 thinks that Player 1 would choose "Up". And Player 1 thinks that Player 2 thinks that Player 1 would choose "Up" because Player 2 thinks that Player 1 thinks that Player 2 would choose "Center". And so on.

1.E.1. Relationship between Nash Equilibrium and Dominance

Strict Domination. Let us look again at the Prisoners' Dilemma Game in Figure 1.4:

		Prisoner 2	
		Cooperate (C)	Defect (D)
Prisoner 1	Cooperate (C)	$(-1, -1)$	$(-9, \underline{0})$
	Defect (D)	$(\underline{0}, -9)$	$(\underline{-6}, \underline{-6})$

Figure 1.20: The Prisoners' Dilemma Game

1. For both Player 1 and Player 2, D strictly dominates C.
2. The Nash equilibrium of the Prisoners' Dilemma Game is (D, D).

Remark 1.17. No *strictly* dominated strategies would be played in a Nash equilibrium. The reason is that a strictly dominated strategy is not a best response to any strategy of the opponent. In particular, it is not a best response to the opponent's strategy in the Nash equilibrium.

Weak Domination. Weak domination is a bit tricky. It is possible for a weakly dominated strategy to appear in a Nash equilibrium. We show by example that a weakly dominated strategy could be played in a Nash equilibrium.

Example 1.E.2. Consider the following game:

		Player 2	
		Left	Right
Player 1	Up	$(\underline{1}, \underline{1})$	$(\underline{0}, 0)$
	Down	$(0, \underline{0})$	$(\underline{0}, \underline{0})$

Figure 1.21: Weak Dominance

1. This game has two equilibria: (Up, Left) and (Down, Right).
2. For Player 1, "Down" is weakly dominated by "Up"; for Player 2, "Right" is weakly dominated by "Left".

That is, we get 2 Nash equilibria. One of the equilibria (Down, Right) involves the play of weakly dominated strategies.

1.E.2. Coordination Game

Investment Game. Consider the following investment game. There are n investors. Each investor could invest either \$0 or \$10. If Investor i invests \$0, then he/she gets \$0. If Investor i invests \$10, then

- if at least 90% of the investors invest, Investor i gets a profit of \$15, or a net profit of \$5;
- if less than 90% of the investors invest, Investor i would lose his/her initial investment \$10.

To look for Nash equilibrium, in principle, we need to look at any possible outcome. For example, 1% of the investors invests and 99% do not. There are infinitely many such combinations. In practice, we **guess and check**. Guess and check is a very useful method in these games where there are many players, but not many strategies per player. There are two Nash equilibria in this game:

1. All investors invest: If all other investors invest, then Investor i's best response is to invest.
2. No investor invests: If all other investors do not invest, then Investor i's best response is not to invest.

Remark 1.18. The equilibrium where all investors invest *Pareto dominates* the equilibrium where no investor invests: every investor is better-off in the first equilibrium.

Remark 1.19. Nash equilibrium is a self-fulfilling outcome.

Remark 1.20. Unlike the Prisoners' Dilemma game, pre-play communication works in the coordination game.

Remark 1.21. This model could help us understand bank runs.

- The good equilibrium: everyone has confidence in the bank and leaves their deposits in the bank. The bank could lend some of the money out on a higher interest rate.
- The bad equilibrium: people lose confidence in the bank and start drawing their deposits out. Then the bank does not have enough cash to cover those deposits and becomes bankrupt.

The Battle of the Sexes.

Example 1.E.3. Consider the Battle of the Sexes game: Alice and Bob are considering going out for the night. While at separate workplaces, Alice and Bob must choose to attend either Opera or Movie without communication. Both of them prefer to be together. But as for the entertainment, Alice prefers Opera whereas Bob prefers Movie. The payoff matrix is given in Figure 1.31.

		Bob	
		Opera	Movie
Alice	Opera	$(2, 1)$	$(0, 0)$
	Movie	$(0, 0)$	$(1, 2)$

Figure 1.22: The Battle of the Sexes

There are two (pure strategy) Nash equilibria: (Opera, Opera) and (Movie, Movie).

Remark 1.22. Unlike the investment game, there is a conflict of interest between the two players in the Battle of the Sexes game.

1.F. Applications

1.F.1. Cournot Model of Duopoly

Game Setup. Let q_1 and q_2 denote the quantities (of a homogeneous product) produced by firms 1 and 2, respectively. Let $P(Q) = a - Q$ be the market-clearing price when the aggregate quantity on the market is $Q = q_1 + q_2$. Assume that the total cost to a firm with quantity q_i is $C_i(q_i) = cq_i$, where $c < a$. That is, there are no fixed costs and the marginal cost is constant at c. Following Cournot, suppose that the firms choose their quantities simultaneously.

Normal-Form Representation.

- Players: Firm 1 and 2;
- Strategies: $q_i \in S_i = [0, \infty)$ for Firm i;
- Payoffs: $\pi_i(q_i, q_j) = q_i[P(q_i + q_j) - c] = q_i[a - (q_i + q_j) - c]$ for Firm i. The other firm is denoted by j.

We have learned three ways to solve the problem.

1. Nash Equilibrium. (q_1^*, q_2^*) forms a Nash equilibrium if, for each firm i,

$$\pi_i(q_i^*, q_j^*) \geq \pi_i(q_i, q_j^*) \text{ for all feasible } q_i \in S_i.$$

Equivalently, q_i^* solves

$$\max_{q_i \in S_i} \pi_i(q_i, q_j^*) = \max_{q_i \in [0,\infty)} q_i[a - (q_i + q_j^*) - c].$$

FOC yields

$$q_i^* = \frac{1}{2}(a - q_j^* - c).$$

For (q_1^*, q_2^*) to be a Nash equilibrium, we have

$$\begin{cases} q_1^* = \frac{1}{2}(a - q_2^* - c); \\ q_2^* = \frac{1}{2}(a - q_1^* - c). \end{cases} \implies (q_1^*, q_2^*) = (\frac{a-c}{3}, \frac{a-c}{3}).$$

The profit of each firm is

$$\pi_i(q_i^*, q_j^*) = \frac{a-c}{3}\left[a - (\frac{a-c}{3} + \frac{a-c}{3}) - c\right] = \frac{(a-c)^2}{9} \tag{1.F.1}$$

2. Best Response Curves. We could also solve for the equilibrium graphically using the best response curves. The two best response curves $\mathrm{BR}_1(q_2)$ and $\mathrm{BR}_2(q_1)$ intersect once at the equilibrium quantity pair $(q_1^*, q_2^*) = (\frac{a-c}{3}, \frac{a-c}{3})$.

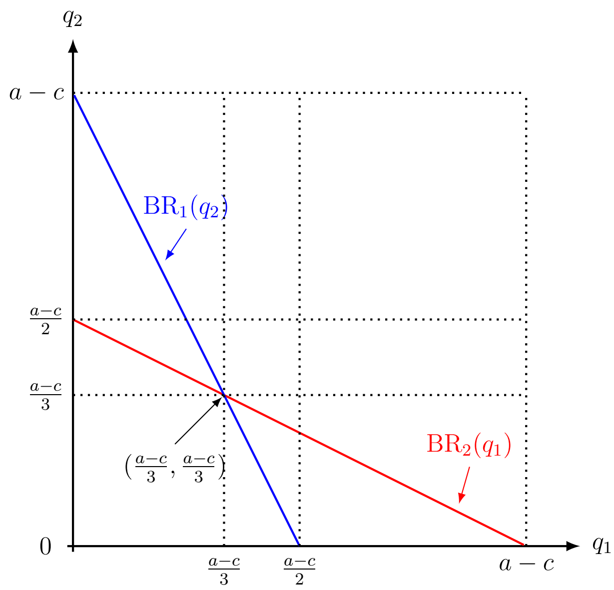

Figure 1.23: Cournot Duopoly

3. Iterated Elimination of Never Best Responses. Similar to our analysis of the partnership game in Section 1.D.3, we can apply iterated elimination of never best responses. In the first round, we eliminate the quantities higher than the monopoly quantity, i.e., $q_i > q_m = \frac{a-c}{2}$, since $q_i > q_m$ is never a best response against any $q_j \geq 0$. In the second round, given that $q \leq q_m = \frac{a-c}{2}$, $q_i < \frac{a-c}{4}$ is never a best response. See Figures 1.24 and 1.25. Repeating the arguments leads to the equilibrium quantity $(q_1^*, q_2^*) = (\frac{a-c}{3}, \frac{a-c}{3})$.

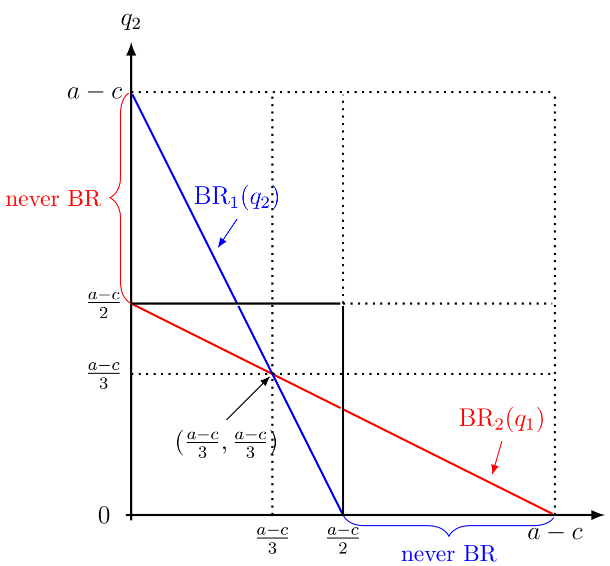

Figure 1.24: First Round Deletion

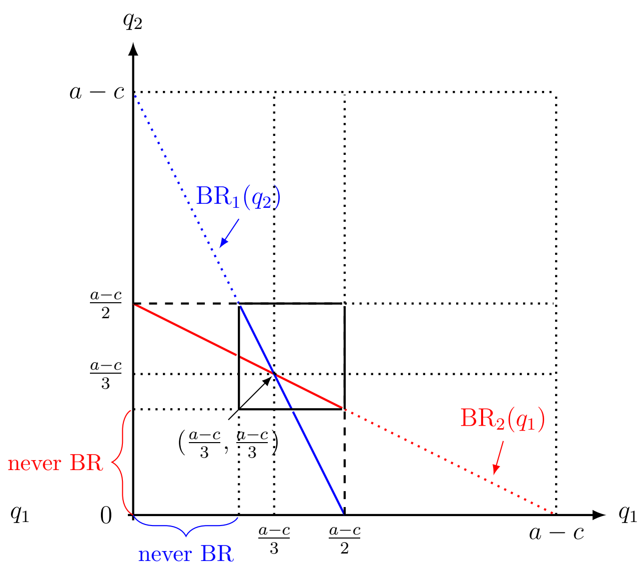

Figure 1.25: Second Round Deletion

The Monopoly Case. Suppose that there is only one firm in the market, the monopolist. The monopolist chooses quantity q to maximize its profit. The marginal cost is still c. Then the monopolist's problem is

$$\max_q q(a-q-c).$$

The solution is

$$q^m = \frac{a-c}{2}.$$

Graphically, the monopoly quantity is where the Marginal Revenue (MR) curve MR $= \frac{\partial q(a-q)}{\partial q} = a - 2q$ and the Marginal Cost (MC) curve MC $= c$, intersect. See Figure 1.26.

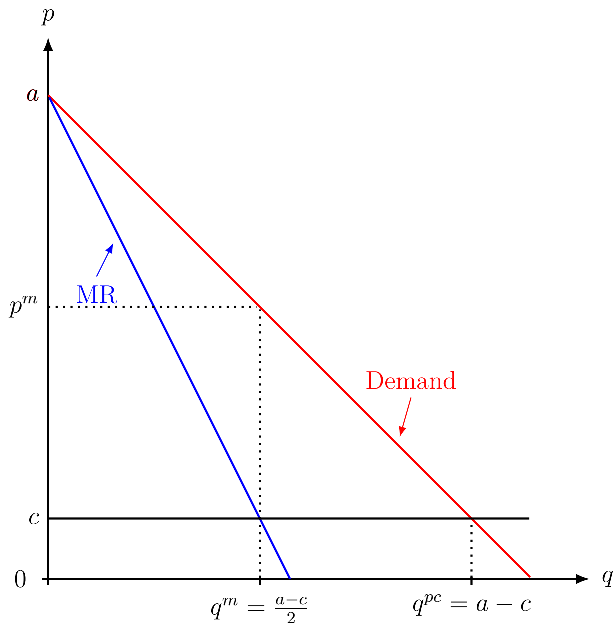

Figure 1.26: Monopoly

Total industry profit for the monopoly case is

$$\pi(q^m) = \frac{a-c}{2}\left[a - \left(\frac{a-c}{2}\right) - c\right] = \frac{(a-c)^2}{4}. \tag{1.F.2}$$

Remark 1.23. The monopoly quantity $q^m = \frac{a-c}{2}$ is smaller than the total quantity $q_1^* + q_2^* = \frac{2(a-c)}{3}$ produced by the Cournot duopoly. In addition, compared to Cournot duopoly, the monopoly price is higher since $p = a - c - Q$, and the total industry profit is higher: $\pi(q^m) = \frac{(a-c)^2}{4} > \frac{2(a-c)^2}{9} = \pi_1(q_1^*, q_2^*) + \pi_2(q_1^*, q_2^*)$.

Question 1.14. Each firm in the Cournot duopoly would be better-off sharing the monopoly profit by each producing half of the monopoly quantity. Why don't the firms do that?

Iterated Elimination May Not Yield a Unique Solution. Consider the three firm version of the Cournot Model. Let Q_{-i} be the sum of quantities of firms other than i. It is still true that any quantity higher than the monopoly quantity $q^m = \frac{a-c}{2}$ is never a best response. In the first round of elimination, we eliminate the quantities $q^i > q^m$ for all firms. After the first round, we have $Q_{-i} \leq q^m + q^m = a - c$. In this case, any $q_i \in [0, \frac{a-c}{2}]$ is a best response to some $Q_{-i} \in [0, a-c]$. Specifically, the best response is $\mathrm{BR}_i(Q_{-i}) = \frac{1}{2}(a - Q_{-i} - c)$. Therefore, we could not further eliminate any quantities. As a result, iterated elimination leads to imprecise prediction in this case.

Remark 1.24. The 3-firm Cournot model could still be solved using Nash equilibrium.

1.F.2. Bertrand Model of Duopoly with Differentiated Products

Game Setup. Let p_1 and p_2 be the prices chosen by firm 1 and 2. Let the quantity consumers demand from firm i be $q_i(p_i, p_j) = a - p_i + bp_j$, where b reflects the extent to which firm i's product is a substitute for firm j's product. Assume $b < 2$. As in the Cournot model, assume that the total cost to a firm with quantity q_i is $C_i(q_i) = cq_i$ where $c < a$, and that the firms act simultaneously.

Normal-Form Representation.

- Players: Firm 1 and 2;
- Strategies: $p_i \in S_i = [0, \infty)$ for Firm i;
- Payoffs: $\pi_i(p_i, p_j) = q_i(p_i, p_j)(p_i - c) = (a - p_i + bp_j)(p_i - c)$.

Nash Equilibrium. (p_i^*, p_j^*) forms a Nash equilibrium if, for each firm i, q_i^* solves

$$\max_{p_i \in [0,\infty)} \pi_i(p_i, p_j^*) = \max_{p_i \in [0,\infty)} (a - p_i + bp_j^*)(p_i - c).$$

The solution to this optimization problem is

$$p_i^* = \frac{1}{2}(a + bp_j^* + c).$$

For (p_1^*, p_2^*) to be a Nash equilibrium, we have

$$\begin{cases} p_1^* = \frac{1}{2}(a + bp_2^* + c); \\ p_2^* = \frac{1}{2}(a + bp_1^* + c). \end{cases} \implies (p_1^*, p_2^*) = \Big(\frac{a+c}{2-b}, \frac{a+c}{2-b}\Big).$$

Best Response Curves. We could also solve for the equilibrium graphically using the best response curves. See Figure 1.27. The two best response curves $\text{BR}_1(p_2)$ and $\text{BR}_2(p_1)$ intersect once at the equilibrium quantity pair $(p_1^*, p_2^*) = (\frac{a+c}{2-b}, \frac{a+c}{2-b})$.

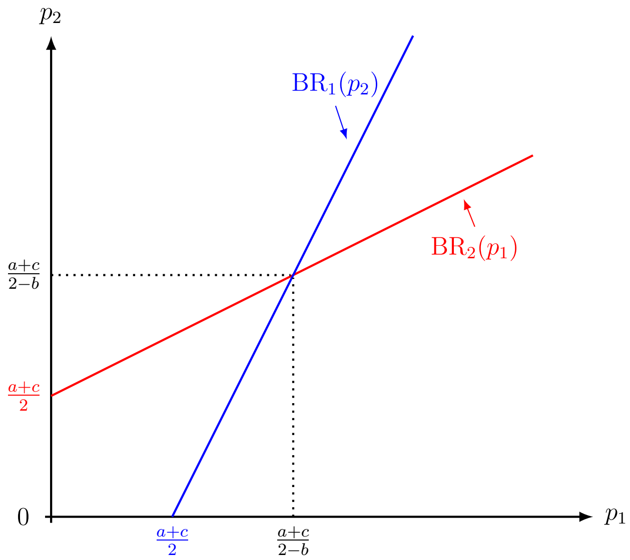

Figure 1.27: Bertrand Duopoly

1.F.3. Candidate-Voter Model

This model is a simplified version of Osborne and Slivinski (1996).

Game Setup. There are n voters, with positions $1, ..., n$. Voters vote for the closest candidate. Unlike the previous voting model in Section 1.C.3:

1. The number of candidates is not fixed.
2. Candidates cannot choose their position. Each voter is a potential candidate.

Normal-Form Representation.

- Players: Voters/Candidates;
- Strategies: to run or not to run;
- Payoffs:
  - Prize if win $= B$, $1/k$ chances of winning if k candidates win;
  - Cost of running $= c$, assuming $B \geq 2c$;
  - If Voter/Candidate i at position x and the winner is at y, then i gets $-|x - y|$.

To better understand the payoff, we look at an example. Suppose that there are 11 voters.

- If Voter/Candidate at position 5 runs and is the sole winner, then he/she gets $B-c$;
- If Voter/Candidate at position 5 runs and Voter/Candidate at position 7 wins, then Voter/Candidate at position 5 gets $-c-2$;
- If Voter/Candidate at position 5 does not run and Voter/Candidate at position 7 wins, then Voter/Candidate at position 5 gets -2.

Nash Equilibrium.

Question 1.15. Is there any NE where no candidate runs?

Answer: No. If no candidate runs, every possible candidate would be better-off running: 0 if not running v.s. $B-c$ if running.

Question 1.16. Is there any NE where one candidate runs?

Answer: Yes if n is odd. Only the center candidate running constitutes a NE:

- For the center candidate, when no other candidate runs, his/her best response is running: 0 if not running v.s. $B-c$ if running.
- For the other candidates, their best responses are not running since they would lose if running and running costs c.

Question 1.17. Is there any NE where two candidates run?

Answer: Yes. The two candidates needs to be of equal distance from the center and cannot be too far apart.
For example, in the 11-Voter/Candidate case, Voter/Candidate 5 and 7 running is a NE. We need to check the following three types of deviations:

- A Voter/Candidate from the outside enters (position $1-4$ and $8-11$): not profitable since i) the entrant would not win the election and ii) the winner would be the candidate farther from him/her for sure.
- The Voter/Candidate in the middle enters (position 6): not profitable since the entrant would not win the election.
- Voter/Candidate 5 and 7 choose not to run: not profitable since running gives $\frac{B}{2}-\frac{1}{2}\cdot 2-c$ whereas not running gives -2. By the assumption $B\geq 2c$, running gives a higher payoff.

If say, Voter/Candidate 1 and 11 runs, then Voter/Candidate 6 would be better-off running. Thus, this would not constitute a NE.

Remark 1.25. There are many NE in this Voter/Candidate Model. And not all of the NE predict candidates "at the center".

1.F.4. Location Model

Game Setup. There are two types of people in the society, namely, Tall (T) and Short (S). The measure of T and S are both 1. There are two towns, namely, East (E) and West (W). Each town could hold measure 1 of people. All people simultaneously choose which town to live in. If more than measure 1 of people chooses one town, then we randomly choose who could stay. The payoff of everyone is the same. Denote a typical person Player i. Player i's utility depends on the fraction of people of his/her own type in his/her town, as shown in Figure 1.28.

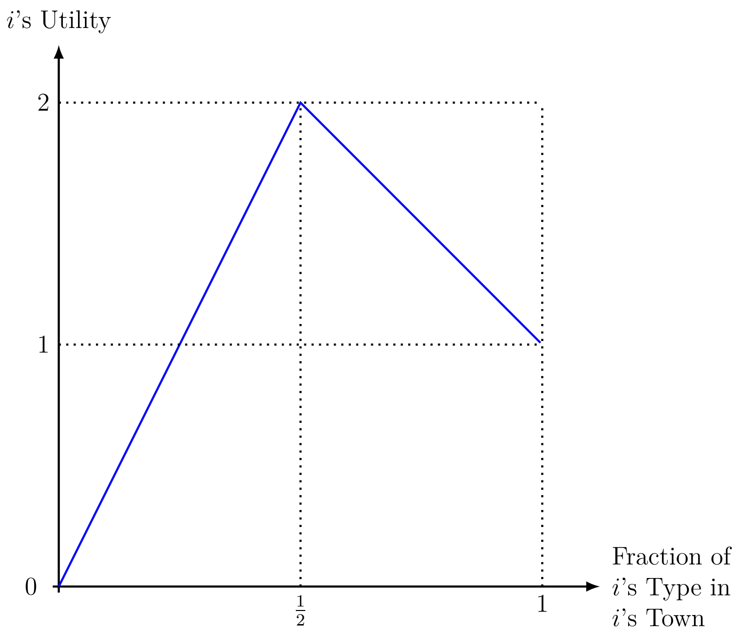

Figure 1.28: Location Model

Nash Equilibrium. There are three Nash Equilibria as follows (Guess and Check):

1. Two Segregated Equilibria
   a) All T in E and All S in W;
   b) All T in W and All S in E;

2. One Integrated Equilibrium: exactly $\frac{1}{2}$ of T and $\frac{1}{2}$ of S in one town.

Remark 1.26. The two segregated equilibria are stable whereas the integrated equilibrium is unstable.

- If starting from 99%/1% (a small deviation away from the segregated equilibrium), the population will eventually restore to the segregated equilibrium;
- If starting from 51%/49% (a small deviation away from the integrated equilibrium), the population will eventually become segregated.

Remark 1.27. Observing segregation does not imply people's preference for segregation. In this location model, everyone prefers integration: an individual obtains 2 in the integrated equilibrium whereas in the segregated equilibrium, he/she only gets 1. However, segregation equilibria exist and are stable. This idea is brought up by Schelling (1971).

Remark 1.28. Actually, according to the description of the game, everyone choosing the same town and getting randomized is also an equilibrium. Integration is attained.

Remark 1.29. The integration outcome could also be obtained via individual randomization. This idea would be clearer after we formally discuss Mixed Strategy Nash Equilibrium in the next section.

1.G. Mixed Strategy Nash Equilibrium

Example 1.G.1. Find the Nash equilibrium in the Rock, Paper, Scissors game.

		Player 2: Rock	Player 2: Paper	Player 2: Scissors
Player 1	Rock	$(0,0)$	$(-1,\underline{1})$	$(\underline{1},-1)$
	Paper	$(\underline{1},-1)$	$(0,0)$	$(-1,\underline{1})$
	Scissors	$(-1,\underline{1})$	$(\underline{1},-1)$	$(0,0)$

Figure 1.29: Rock, Paper, Scissors

Applying our previous definition of Nash Equilibrium, Definition 1.E.1, there exists no (Pure Strategy) Nash Equilibrium in this game.

Question 1.18. How do you play the Rock, Paper, Scissors game?

You probably play each of the actions $\frac{1}{3}$ of the time. We will show below that your and your opponent both playing such a strategy is a Nash Equilibrium after taking into consideration Mixed Strategies.

Definition 1.G.1 (Mixed Strategy)**.** In the normal-form game $G = \{S_1, ..., S_n; u_1, ..., u_n\}$, suppose that Player i has K pure strategies: $S_i = \{s_{i1}, ..., s_{iK}\}$. Then a **mixed strategy** for Player i is a probability distribution $p_i = (p_{i1}, ..., p_{iK})$ over S_i, where $0 \leq p_{ik} \leq 1$ for $k = 1, ..., K$ and $p_{i1} + ... + p_{iK} = 1$.

Remark 1.30. When $p_{ij} = 1$ and $p_{ik} = 0$ for all $k \neq j$, the mixed strategy $p_i = (p_{i1}, ..., p_{iK})$ is the pure strategy s_{ij}.

We could extend the notion of (pure strategy) Nash equilibrium, Definition 1.E.1, to include mixed strategies.

Definition 1.G.2. A mixed strategy profile $(p_1^*, p_2^*, ..., p_n^*)$ is a mixed strategy Nash equilibrium if, for each player i, p_i^* is a best response to p_{-i}^*.

Let us return to the Rock, Paper, Scissors game. Formally, your mixed strategy for the Rock, Paper, Scissors game is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. We show that the strategy profile $((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$ is a NE by showing that $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a best response to the opponent's strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

- The expected payoff from Rock against $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is $\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 1 = 0$;
- The expected payoff from Paper against $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is $\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot (-1) = 0$;
- The expected payoff from Scissors against $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is $\frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 = 0$.

The expected payoff from the mix $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ against $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is thus $\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 0 = 0$. Actually, the expected payoff from any mix $(p, q, 1-p-q)$ against $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is 0. Therefore, playing $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is indeed a best response, albeit weakly. And thus, $\left((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\right)$ is a (mixed strategy) Nash equilibrium.

Question 1.19. Could you check that $\left((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\right)$ is a (mixed strategy) Nash equilibrium for the Matching Pennies game?

		Player 2	
		Head	Tail
Player 1	Head	$(-1, \underline{1})$	$(\underline{1}, -1)$
	Tail	$(\underline{1}, -1)$	$(-1, \underline{1})$

Figure 1.30: Matching Pennies

1.G.1. Finding Mixed Strategy Nash Equilibrium

In the Rock, Paper, Scissors game (and the Matching Pennies game), we had a candidate mixed strategy and checked that it is indeed a mixed strategy Nash equilibrium. Next, we will learn how to find the equilibrium mixing.

Example 1.G.2. Let us revisit the Battle of the Sexes game we saw previously:

		Bob: Opera	Bob: Movie
Alice	Opera	$(2,1)$	$(0,0)$
	Movie	$(0,0)$	$(1,2)$

Figure 1.31: The Battle of the Sexes

Just as what we did in the Rock, Paper, Scissors game, given Alice's and Bob's strategies, we could calculate their payoffs. For example, consider Alice's mixed strategy $p_A = (\frac{1}{5}, \frac{4}{5})$ and Bob's mixed strategy $p_B = (\frac{1}{2}, \frac{1}{2})$.
Then Alice's expected payoffs from Opera and Movie are respectively

$$\mathbb{E}U_A(\text{Opera}, p_B) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1;$$
$$\mathbb{E}U_A(\text{Movie}, p_B) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

Alice's expected payoff from the mixed strategy $p = (\frac{1}{5}, \frac{4}{5})$ is thus

$$\mathbb{E}U_A(p_A, p_B) = \frac{1}{5} \cdot \mathbb{E}U_A(\text{Opera}, p_B) + \frac{4}{5} \cdot \mathbb{E}U_A(\text{Movie}, p_B) = \frac{1}{5} \cdot 1 + \frac{4}{5} \cdot \frac{1}{2} = \frac{3}{5}.$$

We have an important observation:

Observation 1.1. Alice's expected payoff from the mixed strategy p_A is the **weighted average** of the expected payoffs from each of the pure strategies in the mix. And further, the weighted average always lies in-between the lowest and the highest payoffs involved in the mix.

In our example, $\frac{3}{5} = \frac{1}{5} \cdot 1 + \frac{4}{5} \cdot \frac{1}{2} \in [\frac{1}{2}, 1]$. This observation is true in general since

$$\mathbb{E}U_i(p_1, ..., p_n) = \sum_{k=1}^{K} \{p_{ik} \cdot \mathbb{E}U_i(s_{ik}, p_{-i})\};$$

$$\text{and} \quad \sum_{k=1}^{K} \{p_{ik} \cdot \mathbb{E}U_i(s_{ik}, p_{-i})\} \leq \sum_{k=1}^{K} \left\{p_{ik} \cdot \max_{s_{ik}} U_i(s_{ik}, p_{-i})\right\}$$
$$= \sum_{k=1}^{K} \{p_{ik}\} \cdot \max_{s_{ik}} U_i(s_{ik}, p_{-i}) = 1 \cdot \max_{s_{ik}} U_i(s_{ik}, p_{-i}) = \max_{s_{ik}} U_i(s_{ik}, p_{-i});$$
$$\sum_{k=1}^{K} \{p_{ik} \cdot \mathbb{E}U_i(s_{ik}, p_{-i})\} \geq \sum_{k=1}^{K} \left\{p_{ik} \cdot \min_{s_{ik}} U_i(s_{ik}, p_{-i})\right\} = \min_{s_{ik}} U_i(s_{ik}, p_{-i}).$$

An important lesson follows as a consequence of the observation above:

Result 1.1. If a mixed strategy is a best response, then each pure strategy in the mix must be best responses. In particular, each must yield the same expected payoff.

Intuitively, if Player i adopts a mixed strategy, he/she must be indifferent among the pure strategies in the mix. Otherwise, he/she could simply choose the pure strategy with the highest payoff and be better-off.

Applying this idea, we would be able to find the mixed strategy Nash equilibrium for the Battle of the Sexes game. Assume $p_A = (p, 1-p)$ and $p_B = (q, 1-q)$.

Then Alice's expected payoffs from Opera and Movie are respectively

$$\mathbb{E}U_A(\text{Opera}, p_B) = q \cdot 2 + (1-q) \cdot 0 = 2q;$$
$$\mathbb{E}U_A(\text{Movie}, p_B) = q \cdot 0 + (1-q) \cdot 1 = 1-q.$$

According to Result 1.1, for the mixed strategy to be a best response, the two payoffs should be equal:

$$\mathbb{E}U_A(\text{Opera}, p_B) = \mathbb{E}U_A(\text{Movie}, p_B) \implies 2q = 1-q \implies q = \frac{1}{3}.$$

Similarly, for Bob

$$\begin{cases} \mathbb{E}U_B(p_A, \text{Opera}) = p \cdot 1 + (1-p) \cdot 0 = p \\ \mathbb{E}U_B(p_A, \text{Movie}) = p \cdot 0 + (1-p) \cdot 2 = 2 - 2p \end{cases}$$
$$\mathbb{E}U_B(p_A, \text{Opera}) = \mathbb{E}U_B(p_A, \text{Movie}) \implies p = 2-2p \implies p = \frac{2}{3}.$$

The mixed strategy Nash equilibrium is $\left(p_A = (\frac{2}{3}, \frac{1}{3}), p_B = (\frac{1}{3}, \frac{2}{3})\right)$.

Remark 1.31. Notice that using Alice's payoff and applying the indifferent condition, we solve for Bob's mixing, i.e., $p_B = (q, 1-q)$, and similarly, using Bob's payoff, we solve for Alice's mixing, i.e., $p_A = (p, 1-p)$.

Remark 1.32. A maybe easier to remember version of Remark 1.31: Bob's equilibrium mix makes Alice indifferent and Alice's equilibrium mix make Bob indifferent.

To check whether the mixed strategy is indeed a Nash equilibrium, previously for the Rock, Paper, Scissors game, we checked that there is no strictly profitable deviation to all pure strategies and all other possible mixed strategies. Actually, applying Observation 1.1, checking all pure strategies are sufficient.

Question 1.20. Why is it that checking all pure strategies are sufficient?

Suppose there isn't any profitable pure-strategy deviation. Then there can't be any profitable mixed-strategy deviation, because the highest expected return a player could ever get from a mixed strategy is one of the pure strategies in the mix.

1.G.2. Other Examples

Tax Paying Game. Consider the following game between the auditor and taxpayer:

		Taxpayer: Honest	Taxpayer: Cheat
Auditor	Audit	$(2, 0)$	$(4, -10)$
	Not Audit	$(4, 0)$	$(0, 4)$

Figure 1.32: Tax Paying Game

Let us first look for pure strategy Nash equilibrium.

		Taxpayer: Honest (H)	Taxpayer: Cheat (C)
Auditor	Audit (A)	$(2, \underline{0})$	$(\underline{4}, -10)$
	Not Audit (N)	$(\underline{4}, 0)$	$(0, \underline{4})$

Figure 1.33: Paying Taxes

There is no pure strategy Nash equilibrium in this game.
Next, we look for mixed strategy Nash equilibrium. Let the auditor's mixed strategy be $p_A = (p, 1-p)$ and the taxpayer's mixed strategy be $p_T = (q, 1-q)$.

We use the auditor's payoff to find $(q, 1-q)$. Actions A and N give the auditor the following expected payoffs respectively

$$\begin{cases} \mathbb{E}U_A(A, p_T) = 2q + 4(1-q) = 4 - 2q \\ \mathbb{E}U_A(N, p_T) = 4q + 0 \cdot (1-q) = 4q. \end{cases}$$

Auditor is indifferent between A and N:

$$\mathbb{E}U_A(A, p_T) = \mathbb{E}U_A(N, p_T) \implies 4 - 2q = 4q \implies q = \frac{2}{3}.$$

Similarly, we use the taxpayer's payoff to find $(p, 1-p)$. Actions H and C give the taxpayer the following expected payoffs respectively

$$\begin{cases} \mathbb{E}U_T(p_A, H) = 0 \cdot p + 0 \cdot (1-p) = 0 \\ \mathbb{E}U_T(p_A, C) = -10p + 4(1-p) = 4 - 14p. \end{cases}$$

Taxpayer is indifferent between H and C:

$$\mathbb{E}U_T(p_A, H) = U_T(p_A, C) \implies 0 = 4 - 14p \implies p = \frac{2}{7}.$$

Therefore, the mixed strategy Nash equilibrium is $\big(p_A = (\frac{2}{7}, \frac{5}{7}), p_T = (\frac{2}{3}, \frac{1}{3})\big)$.

Question 1.21. What if we raise the fine from -10 to -20. Will this policy raise compliance rate q?

Answer: No. Calculation gives $q = \frac{2}{3}$ and $p = \frac{1}{6}$. The effect is lower audit rate $(\frac{1}{6} < \frac{2}{7})$.

Question 1.22. What policies would raise tax compliance rate q?

Answer: We need to change the payoffs of the auditor. For example, making it less costly to do an audit, or giving the auditors a bigger gain for catching a cheater.

Penalty Kick Game. Let us reconsider the Penalty Kick Game in Section 1.D.2.

		Goalkeeper: Left (L)	Goalkeeper: Right (R)
Shooter	left (l)	$(4, -4)$	$(9, -9)$
	middle (m)	$(6, -6)$	$(6, -6)$
	right (r)	$(9, -9)$	$(4, -4)$

Figure 1.34: Penalty Kick Game

Let Shooter's mixed strategy be $p_S = (p_1, p_2, 1 - p_1 - p_2)$ and the Goalkeeper's mixed strategy be $p_G = (q, 1 - q)$. If Shooter mixes between the three actions, then

$$4q + 9(1 - q) = 6q + 6(1 - q) = 9q + 4(1 - q).$$

It is not hard to check that there is no q satisfying the above equation.

Next, we consider Shooter mixing between two strategies.

1. Suppose that Shooter mixes between l and m, then the indifference condition gives

$$4q + 9(1 - q) = 6q + 6(1 - q) \implies q = \frac{3}{5}.$$

   In this case, the payoff from strategy l and m are both 6, whereas the payoff from strategy r is $9 \cdot \frac{3}{5} + 4 \cdot \frac{2}{5} = 7 > 6$. Therefore, a deviation to r is profitable. This cannot be an equilibrium.

2. Similarly, Shooter mixing between m and r is not an equilibrium.
3. Suppose that Shooter mixes between l and r, then the indifference condition gives

$$4q + 9(1 - q) = 9q + 4(1 - q) \implies q = \frac{1}{2}.$$

   In this case, the payoff from strategy l and r are both $9 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} = \frac{13}{2}$, whereas the payoff from strategy m is $6 < \frac{13}{2}$. Therefore, a deviation to m is not profitable. This can be an equilibrium. In this equilibrium, $p_2 = 0$. Then we use Goalkeeper's payoff to calculate Shooter's mixed strategy:

$$(-4)p_1 + (-9)(1 - p_1) = (-9)(p_1) + (-4)(1 - p_1) \implies p_1 = \frac{1}{2}.$$

Therefore, the mixed strategy Nash equilibrium is $\big(p_S = (\frac{1}{2}, 0, \frac{1}{2}), p_G = (\frac{1}{2}, \frac{1}{2})\big)$.

Remark 1.33. Recall that when we first study the Penalty Kick Game in Section 1.D.2, we have shown that m is never a best response to any belief. Therefore, it should not be surprising that in the mixed strategy equilibrium, m is not played.

1.G.3. Dominance and Best Responses

Recall that if a strategy s_i is strictly dominated, then there is no belief that Player i could hold (about the other players' strategies) such that it would be optimal to play s_i. The converse is also true for the 2-player case, provided that we allow for mixed strategies:

if there is no belief that Player i could hold (about the other players' strategies) such that it would be optimal to play the strategy s_i, then there exists another strategy that strictly dominates s_i.[4]

We will illustrate the second result in an example. Consider the Penalty Kick Game. In the Penalty Kick Game, we know that m is never a best response for Shooter. Now we show that there exists a (mixed) strategy that strictly dominates m. Let such a strategy be $(p, 0, 1-p)$.

Playing against Goalkeeper's strategy $(q, 1-q)$ for $q \in [0,1]$,

1. m gives 6;
2. the mixed strategy $(p, 0, 1-p)$ gives $4pq + 9p(1-q) + 9(1-p)q + 4(1-p)(1-q) = (-10p+5)q + 5p + 4$.
3. For the mixed strategy to dominate m for any $q \in [0,1]$, we need $5p + 4 > 6$ (when $q = 0$) and $-10p + 5p + 9 > 6$ (when $q = 1$). Therefore, any mix with $p \in (\frac{2}{5}, \frac{3}{5})$ will do.

1.G.4. Interpretations of Mixed Strategies

1. People literally randomizing: for example, the players in the rock, paper and scissors game and the players in the penalty kick game.
2. Beliefs of others' actions: for example, in the battle of the sexes game, we could think about Alice's mixture as being a statement about what Bob believes that Alice is going to do. Holding such a belief, Bob is indifferent between the two actions.
3. Proportion of players: for example, the taxpayer's strategy $(\frac{2}{3}, \frac{1}{3})$ in the tax paying game can be thought of as the proportion of taxpayers being honest $(\frac{2}{3})$ and cheating $(\frac{1}{3})$ on their taxes respectively.

[4] This result is proven by Pearce (1984). And note that for the result to be true for the n-player case, we need to allow for correlated strategies.