Chapter 1. Static Games of Complete Information

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1.A. What is Game Theory?

Game theory is a method of studying strategic situations.

Strategic situations are settings where the outcomes that affect you depend on your own actions and the actions of others.

What is Game Theory?

Example 1.A.1.

i. A monopolist is non-strategic. There are no competitors.
ii. Firms under perfect competition are non-strategic. Prices are taken as given, so firms do not need to worry about the actions of the competitors.
iii. Oligopolists are strategic. The actions of the firms affect one another.

Game theory applies in economics, laws, biology, sports, etc. We will discuss some applications in this course.

1.B. Normal-Form Representation of Games

Let us consider the Prisoners' Dilemma Game.

- If neither prisoner confesses then both will be convicted of a minor offense and sentenced to one month in jail.
- If both confess then both will be sentenced to jail for six months.
- If one confesses but the other does not, then the confessor will be released immediately but the other will be sentenced to nine months in jail – six for the crime and a further three for obstructing justice.

Prisoners' Dilemma Game

- We label the two suspects Prisoner 1 and Prisoner 2. They are the players of the game.
- The strategies each of the prisoners could take are
    1. "Not confess" or Cooperate, denoted by C,
    2. "Confess" or Defect, denoted by D.

Outcome matrix for Prisoner 1

The information on the outcomes could be concisely recorded in the tables.

		Prisoner 2	
		C	D
Prisoner 1	C	1 month in prison	9 months in prison
	D	released	6 months in prison

Prisoner 1's Outcome

Outcome matrix for Prisoner 2

		Prisoner 2	
		C	D
Prisoner 1	C	1 month in prison	released
	D	9 months in prison	6 months in prison

Prisoner 2's Outcome

Outcome matrix

Rather than drawing two tables, we could super-impose the second table on top of the first table, forming the outcome matrix.

		P2	
		C	D
P1	C	$(1$ month, 1 month$)$	$(9$ months, released$)$
	D	(released, 9 months$)$	$(6$ months, 6 months$)$

Outcome Matrix

Prisoners' Dilemma Game

Question. What would you do if you were one of the prisoners?

Prisoners' Dilemma Game

To analyze the game, we are still missing the payoffs.

That is, we need to know what the players care about.

- In this prisoners' dilemma game, we assume that the prisoners care about their own jail time, and they get utility -1 for 1 month in prison.

Based on the outcome matrix, we could write down the payoff matrix for the prisoners' dilemma game.

Payoff Matrix

		Prisoner 2	
		C	D
Prisoner 1	C	$(-1, -1)$	$(-9, 0)$
	D	$(0, -9)$	$(-6, -6)$

Payoff Matrix

The payoff matrix contains all the information we need to analyze the prisoners' dilemma game.

It is the normal-form representation of the game.

Normal-form Representation of a Game

Formally, the normal-form representation of a game specifies:

1. the players in the game,
2. the strategies available to each player,
3. the payoffs received by each player for each combination of strategies that could be chosen by the players.

Notations

	Notations	in PD Game
Players	Player i for $i = 1, .., n$	Prisoner 1 and 2
Strategies	S_i: i's strategy space (set of strategies of Player i)	$\{C, D\}$ for $i = 1, 2$
	s_i: a strategy for Player i	C or D for $i = 1, 2$s
	$s = (s_1, ...s_i, ..., s_N)$: a strategy profile (a play of the game)	e.g. (C, C)
Payoffs	$u_i(s) = u_i(s_1, ...s_i, ..., s_N)$	e.g. $u_1(C, C) = -1$

Normal-form Representation of a Game

Definition 1.B.1. The normal-form representation of an n-player game specifies the players' strategy space $S_1, ..., S_n$ and their payoff functions $u_1, ..., u_n$. We denote this game by $G = \{S_1, ..., S_n; u_1, ..., u_n\}$.

Remark: Timing vs. Information

To analyze a game, information (what does Player i know) is more important than timing (when do the players move).

- For example, in the prisoners' dilemma game, the prisoners do not need to move simultaneously, it suffices that each prisoner choose his/her action without knowledge of the other's choices. This point will be addressed later in the course.

1.C. Iterated Elimination of Strictly Dominated Strategies

To solve the prisoners' dilemma game, we will use the idea that a rational player will not play a strictly dominated strategy.

Definition 1.C.1. Strategy s_i' is strictly dominated by strategy s_i'' if:

$$u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i}) \text{ for all } s_{-i},$$

where $s_{-i} = (s_1, ...s_{i-1}, s_{i+1}, ..., s_n)$ denotes a strategy profile of all other players except i.

Strict Domination

- Definition 1.C.1 tells us that strategy s_i' is strictly dominated by strategy s_i'' if the payoff from strategy s_i'' is strictly higher than that from strategy s_i' regardless of the other players' choices.
- We also say strategy s_i'' strictly dominates strategy s_i'.

1.C.1. The Prisoners' Dilemma Game

We apply the idea "a rational player will not play a strictly dominated strategy" to PD game. For P1,

- If P2 chooses C, then C yields -1 and D yields 0;
- If P2 chooses D, then C yields -9 and D yields -6.

In either case, D is strictly better for P1.

The Prisoners' Dilemma Game

- P1's strategy C is strictly dominated by strategy D, or P1's strategy D strictly dominates strategy C.
- P1 should choose D.
- Following the same logic, P2 should also choose D.
- Thus, (D, D) will be the solution of the game.

The Prisoners' Dilemma Game

Remark. The process is called elimination of strictly dominated strategies.

Remark. The only individually rational solution (D, D) is Pareto inefficient. That is, both prisoners would obtain higher payoffs if they choose (C, C).

The Prisoners' Dilemma Game

The prisoners' dilemma game has many applications, including

- arms race (D: high level of arms, C: low level of arms);
- price wars (D: undercut price, C: set high price);
- free-rider problem in the provision of public goods
- joint project (D: shirk, C: cooperate).

The Prisoners' Dilemma Game

Question. Can you think of any ways to make the good outcome (C, C) happen?

The Prisoners' Dilemma Game

Question. Notice that direct communication between the players would not work. Why?

1.C.2. Other Examples

Example 1.C.1. Consider the following game:

		Player 2	
		Left	Right
Player 1	Up	$(1, 0)$	$(1, 2)$
	Down	$(0, 3)$	$(0, 1)$

- Players: Player 1 and Player 2;
- Strategy spaces: $S_1 = \{$Up, Down$\}$, $S_2 = \{$Left, Right$\}$.
- Payoffs: e.g. $u_1($Up, Left$) = 1$, $u_2($Up, Left$) = 0$.

Example 1.C.1

Question. Is there any strategy that is strictly dominated?

Example 1.C.1

For Player 1,

- "Up" strictly dominates "Down".
- Rational Player 1 would not choose "Down".

For Player 2,

- "Right" is better than "Left" if Player 1 plays "Up";
- "Left" is better than "Right" if Player 1 plays "Down".

Example 1.C.1

Question. What would Player 2 do?

Example 1.C.1

If Player 2 knows that Player 1 is rational, then Player 2 could eliminate "Down" from Player 1's strategy space.

		Player 2	
		Left	Right
Player 1	Up	$(1, 0)$	$(1, 2)$

Then, rational Player 2 would choose "Right".

Summing up, the solution of the game is (Up, Right).

Example 1.C.1

Remark. Even though neither "Left" or "Right" is strictly dominated for Player 2, by figuring out what Player 1 would do, Player 2 would choose "Right" (as long as Player 2 is rational and Player 2 knows that Player 1 is rational).

Example 1.C.2

Example 1.C.2. Consider the following game:

		Player 2		
		Left	Middle	Right
Player 1	Up	$(1, 0)$	$(0, 1)$	$(1, 2)$
	Down	$(0, 3)$	$(2, 0)$	$(0, 1)$

Example 1.C.2

For Player 1, neither "Up" or "Down" is strictly dominated:

- "Up" is better than "Down" if Player 2 plays "Left";
- "Down" is better than "Up" if Player 2 plays "Middle".

For Player 2, "Middle" is strictly dominated by "Right":

- "Middle" is better than "Right" no matter whether Player 1 plays "Up" or "Down".

So, rational Player 2 would not play "Middle".

Example 1.C.2

- Thus, if P1 knows that P2 is rational, then P1 could eliminate “Middle” from P2’s strategy space. The game becomes the one in Example 1.C.1.
- Then, if P1 is rational (and P1 knows that P2 is rational) then P1 will not play “Down”.
- Thus, if P2 knows that P1 is rational, and P2 knows that P1 knows that P2 is rational, then P2 can eliminate “Down” from P1’s strategy space.
- Then a rational P2 would choose “Right”.
- Therefore, the solution of the game is (Up, Right).

Iterated elimination of strictly dominated strategies

Remark. The process is called iterated elimination of strictly dominated strategies.

Remark. Iterated elimination of strictly dominated strategies has stronger predictive power.

1.C.3. Application of Iterated Elimination of Strictly Dominated Strategeis: Voting

- 2 candidates choose political positions for an election.
- 10 positions to choose from: 1 to 10.
- Voters uniformly distributed: 10% at each position.
- Voters will vote for the closest candidate.
- If there is a tie, voters of that position split evenly.
- Candidates' objective is to maximize the share of votes.

Voting

Question. Who are the players? What are the strategy spaces and payoffs?

Voting

Question. Suppose that one of the candidate is at position 2 and the other at position 6. What are their shares of votes?

Voting: Analysis

We solve the game using Iterated Elimination of Strictly Dominated Strategies.

Question. Does position 2 dominate position 1?

We need to work out the share of votes a candidate would get if she chooses position 1 or position 2, against all different positions the other candidate could choose.

Voting: Analysis

Question. Does position 3 dominate position 2?

Voting: Analysis

Question. What if positions 1 and 10 are deleted (since they are strictly dominated by position 2 and position 9 respectively)? Does position 3 dominate position 2 then?

We need to check the share of votes Candidate 1 would get if he/she chooses position 2 or position 3, against all different positions except position 1 and 10.

Voting: Analysis

- We delete positions 2 and 9 since they are strictly dominated (after we delete positions 1 and 10).
- Then we could iterate once more and delete positions 3 and 8.
- And after that, we could delete positions 4 and 7.
- In the end, we are left with positions 5 and 6.
- So the prediction from our analysis is that the candidates would both choose the center positions.

Voting

Remark. In political science, it is called the Median Voter Theorem.

Remark. This idea was introduced by Downs (1957) in political science. Hotelling (1929) raised a similar idea in economics on product positioning.

1.C.4. Drawbacks of Iterated Elimination of Strictly Dominated Strategies

1. To do the iterated elimination, we require a further assumption on what the players know about each other's rationality;
2. The process is applicable to only a small fraction of games.

Common Knowledge of Rationality

- Iterated Elimination requires further assumptions on what the players know about each other's rationality.
- To iterate an arbitrary number of rounds, we need to assume not only that all the players are rational, but also that all the players know that all the players are rational, and that all the players know that all the players know that all the players are rational, and so on, ad infinitum. This is called common knowledge of rationality.

Common Knowledge vs. Mutual Knowledge

An event is mutual knowledge if everyone knows it.

Example 1.C.3 (The Hat Puzzle)**.** Two individuals wear hats of two possible colors: black or white. Each individual observes the color of the other individual's hat but not the color of his own hat. Suppose that both of them wear a white hat.

- Situation 1: An outsider says "I will count slowly. Raise your hand if you know the color of your hat".
- Situation 2: Before counting, the outsider mentions "At least one of you wears a white hat".

Common Knowledge vs. Mutual Knowledge

- Situation 1: No one raises their hands and the counts go on forever.
- Situation 2: Both players raise their hands in round 2.

"At least one of you wears a white hat" is mutual knowledge in situation 1 and becomes common knowledge in situation 2.

Prediction Power

Applying Iterated Elimination of Strictly Dominated Strategies, we are only able to solve a limited number of games.

Example 1.C.4. The following game has no strictly dominated strategies.

		Player 2 Left	Player 2 Right
Player 1	Up	$(5, 1)$	$(0, 2)$
Player 1	Middle	$(1, 3)$	$(4, 1)$
Player 1	Down	$(4, 2)$	$(2, 3)$

1.C.5. Weakly Dominated Strategies

Similar to strictly dominated strategies (Definition 1.C.1), we define weakly dominated strategies as follows.

Definition 1.C.2. Strategy s_i' is weakly dominated by strategy s_i'' if

$$u_i(s_i', s_{-i}) \leq u_i(s_i'', s_{-i}) \text{ for all } s_{-i};$$

$$u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i}) \text{ for some } s_{-i}.$$

where $s_{-i} = (s_1, ...s_{i-1}, s_{i+1}, ..., s_n)$ denotes a strategy profile of all other players except i.

Weakly Dominated Strategies

Definition 1.C.2 tells us that strategy s_i' is weakly dominated by strategy s_i'' if the payoff from strategy s_i'' is

- weakly higher than that from strategy s_i' for all of the other players' choices and
- strictly higher for some of the other players' choices.

We also say strategy s_i'' weakly dominates strategy s_i'.

Weakly Dominated Strategies

Question. Could we have "iterated elimination of weakly dominated strategies"?

The problem with iterative elimination of weakly dominated strategy is that the prediction may depend on the order in which actions are eliminated.

Weakly Dominated Strategies

Example 1.C.5. Consider the following game:

		Player 2		
		Left	Middle	Right
Player 1	Up	$(0,1)$	$(1,0)$	$(0,0)$
	Down	$(0,0)$	$(0,0)$	$(1,0)$

Figure 1.1: Example 1.C.5

For P2, "Middle" and "Right" are weakly dominated by "Left".

Example 1.C.5

If only "Right" is removed,

	Left	Middle
Up	$(0,1)$	$(1,0)$
Down	$(0,0)$	$(0,0)$

$\xrightarrow[\text{dominated by U}]{\text{D weakly}}$

	Left	Middle
Up	$(0,1)$	$(1,0)$

$\xrightarrow[\text{dominated by L}]{\text{M strictly}}$

	Left
Up	$(0,1)$

Example 1.C.5

If only "Middle" is removed,

	Left	Right
Up	$(0,1)$	$(0,0)$
Down	$(0,0)$	$(1,0)$

$$\xRightarrow[\text{dominated by D}]{\text{U weakly}}$$

	Left	Right
Down	$(0,0)$	$(1,0)$

Example 1.C.5

If both “Right” and “Middle” are removed,

the prediction is

	Left
Up	$(0, 1)$
Down	$(0, 0)$

Let's Play a Game!

Example 1.C.6 (Guessing Game)**.**

- Everyone in the class pick an integer from $[1, 100]$.
- The winner is the person whose number is closest to two-thirds times the average in the class.

1.C.6. Application of Weakly Dominated Strategies: Second-Price Auction

- One indivisible good for sale
- Valuations of N potential buyers are independently drawn from a uniform distribution with support $[0, 1]$.
- Denote Buyer i's valuation by v_i.

Second-Price Auction

Auction rule:

- Buyers bid simultaneously: $b_i \in [0, +\infty)$.
- Bidder with the highest bid wins the auction and pays the second highest bid.
- If k buyers submit the same highest bid, then each of the k buyers has $1/k$ chance of winning the good. The payment is the highest bid (since there is a tie).

Second-Price Auction

Question. How will you bid?

Second-Price Auction: Analysis

Buyer i 's payoff when submitting the bid b_i is

$$u_i = \begin{cases} 0 & \text{if } b_i < \max_{j \neq i} b_j \\ \frac{v_i - \max_{j \neq i} b_j}{k} & \text{if } b_i = \max_{j \neq i} b_j \\ v_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \end{cases}$$

where k is the number of buyers bidding b_i when $b_i = \max_{j \neq i} b_j$.

Second-Price Auction: Analysis

Claim. $b_i \neq v_i$ is weakly dominated by $b_i = v_i$.

Second-Price Auction: Analysis

	bidding $b_i'' < v_i$	bidding $b_i = v_i$	bidding $b_i' > v_i$
$m > b_i'$	0	0	0
$m = b_i'$	0	0	$-(m - v_i)/k$
$m \in (v_i, b_i')$	0	0	$-(m - v_i)$
$m = v_i'$	0	0	0
$m \in (b_i'', v_i)$	0	$v_i - m$	$v_i - m$
$m = b_i''$	$(v_i - m)/k$	$v_i - m$	$v_i - m$
$m < b_i''$	$v_i - m$	$v_i - m$	$v_i - m$

where $m = \max_{j \neq i} b_j$.

1.D. Best Responses

1.D.1. Example 1.C.4 Revisited

		Player 2	
		Left	Right
Player 1	Up	$(5, 1)$	$(0, 2)$
	Middle	$(1, 3)$	$(4, 1)$
	Down	$(4, 2)$	$(2, 3)$

We already know that none of the strategies are strictly dominated.

Example 1.C.4

Question. Suppose that you are Player 1.

- Could you justify the behavior of choosing "Up"?
- How about "Middle" and "Down"?

Example 1.C.4

To answer this question, we need to consider your belief on P2's strategy.

- Let your belief on the probability that P2 would choose "Right" be $\Pr(\text{Right}) = p_r$.
- Then you belief on the probability that Player 2 would choose "Left" is $1 - p_r$.

Example 1.C.4

"Up" dominates "Middle" when

$$5 \cdot (1 - p_r) + 0 \cdot p_r \geq 1 \cdot (1 - p_r) + 4 \cdot p_r \implies p_r \leq \frac{1}{2};$$

"Up" dominates "Down" when

$$5 \cdot (1 - p_r) + 0 \cdot p_r \geq 4 \cdot (1 - p_r) + 2 \cdot p_r \implies p_r \leq \frac{1}{3}.$$

- "Up" dominates both "Middle","Down" when $p_r \leq \frac{1}{3}$.
- The belief $p_r \leq \frac{1}{3}$ justifies the choice of "Up".
- Formally, "Up" is called a Best Response (BR) to the belief $p_r \leq \frac{1}{3}$.

Best Response

Definition 1.D.1. Player i's strategy $\hat{s}_i$ is a Best Response (BR) to the belief p about the other players' choices if

$$\mathbb{E}u_i(\hat{s}_i, p) \geq \mathbb{E}u_i(s_i', p) \text{ for all } s_i' \in S_i,$$

or $\hat{s}_i$ solves

$$\max_{s_i} \mathbb{E}u_i(s_i, p).$$

Best Response

Remark. Definition 1.D.1 does not limit to two-player games.

Besides, each player could have any number of strategies.

Example 1.C.4

Question. Under which belief is "Down" a Best Response? How about "Middle"?

Example 1.C.4

Dominance relationships could be more clearly shown in a figure.

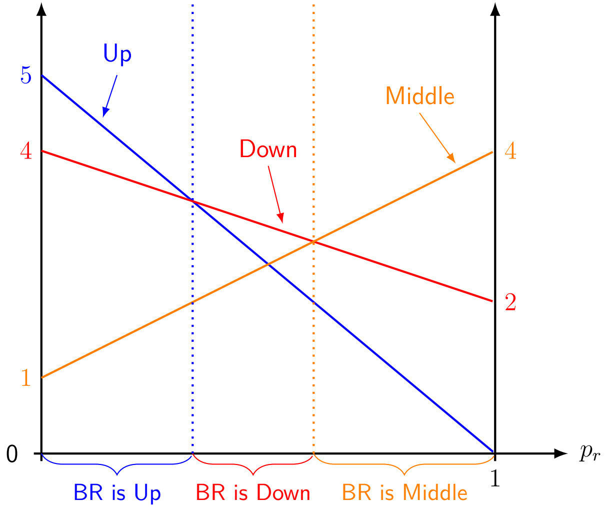
Player 1's expected payoffs from each strategy:

$$\mathbb{E}u_1(\mathsf{Up}, p_r) = 5 \cdot (1 - p_r) + 0 \cdot p_r = 5 - 5p_r$$

$$\mathbb{E}u_1(\mathsf{Middle}, p_r) = 1 \cdot (1 - p_r) + 4 \cdot p_r = 1 + 3p_r$$

$$\mathbb{E}u_1(\mathsf{Down}, p_r) = 4 \cdot (1 - p_r) + 2 \cdot p_r = 4 - 2p_r$$

Example 1.C.4



1.D.2. Penalty Kick Game

Example 1.D.1 (Penalty Kick Game)**.** Consider the following penalty kick game.

		Goalkeeper	
		Left (L)	Right (R)
Shooter	left (l)	$(4, -4)$	$(9, -9)$
	middle (m)	$(6, -6)$	$(6, -6)$
	right (r)	$(9, -9)$	$(4, -4)$

Penalty Kick Game

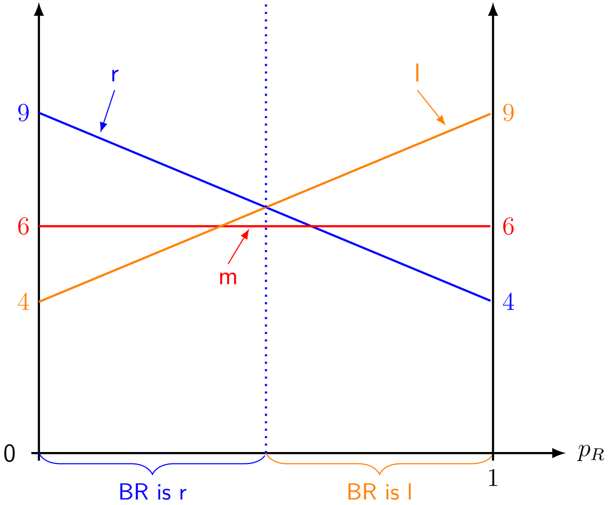
The expected payoffs:

$$\mathbb{E}u_S(l, p_R) = 4 \cdot (1 - p_R) + 9 \cdot p_R = 4 + 5p_R$$

$$\mathbb{E}u_S(m, p_R) = 6 \cdot (1 - p_R) + 6 \cdot p_R = 6$$

$$\mathbb{E}u_S(r, p_R) = 9 \cdot (1 - p_R) + 4 \cdot p_R = 9 - 5p_R$$

Penalty Kick Game



Penalty Kick Game

It is clear from the figure that m is not a best response to any belief. Therefore, the shooter should not kick to the middle.

Remark. One should not choose a strategy that is never a Best Response (BR) to any belief.

1.D.3. Partnership Game

We will learn to apply the idea that "players do not choose a strategy that is never a BR" in the following partnership game.

- 2 agents form a partnership.
- For partnership to work, each agent $i = 1, 2$ needs to put in effort $s_i \in S_i = [0, 4]$.
- Cost of effort is $-s_i^2$.
- Total profit from partnership is $4(s_1 + s_2 + bs_1 s_2)$.
- Agents share profit equally, each obtaining 50%.

Partnership Game: Analysis

Payoffs for the 2 agents are

$$u_1(s_1, s_2) = \frac{1}{2}[4(s_1 + s_2 + bs_1 s_2)] - s_1^2;$$
$$u_2(s_1, s_2) = \frac{1}{2}[4(s_1 + s_2 + bs_1 s_2)] - s_2^2.$$

Best responses $\hat{s}_1$ and $\hat{s}_2$ solves

$$\max_{s_1} 2(s_1 + s_2 + bs_1 s_2) - s_1^2;$$

$$\max_{s_2} 2(s_1 + s_2 + bs_1 s_2) - s_2^2.$$

Partnership Game: Analysis

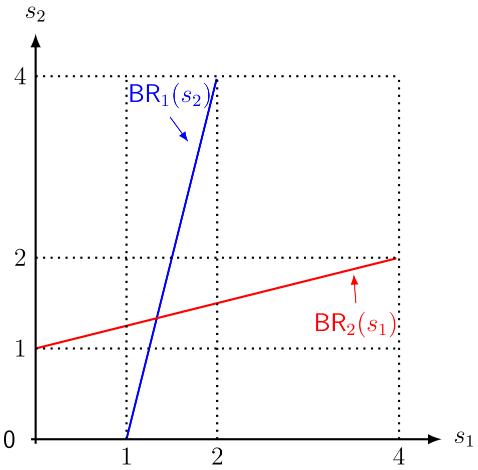
First Order Condition (FOC) gives

$$\hat{s}_1 = 1 + bs_2 = \mathsf{BR}_1(s_2);$$

$$\hat{s}_2 = 1 + bs_1 = \mathsf{BR}_2(s_1).$$

Partnership Game: Analysis

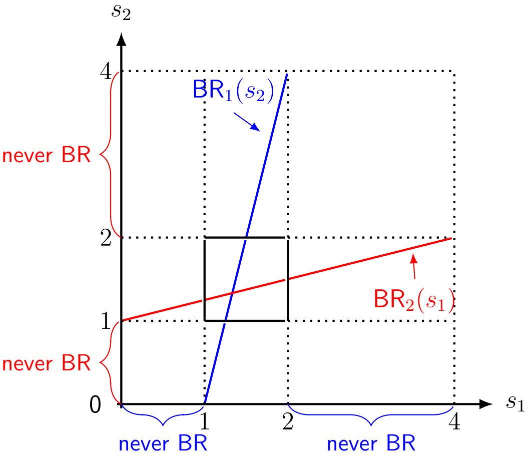
The following graph illustrate the case with $b = \frac{1}{4}$.



Partnership Game: Analysis

For each agent, $s_i < 1$ and $s_i > 2$ are never best responses.

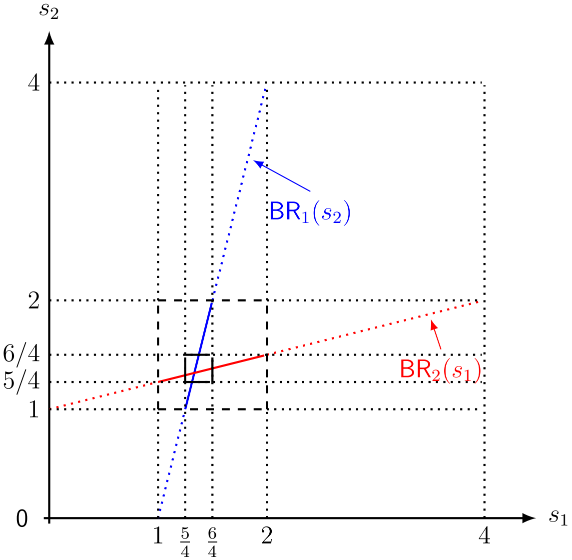
Delete these strategies.



Partnership Game: Analysis

- We further delete those strategies that were never best responses to the opponent's strategies after the first round of deletion.
- This procedure is similar to the Iterated Elimination of Strictly Dominated Strategies we learned before.

Partnership Game: Analysis



Partnership Game: Analysis

- Again, we could delete all strategies that were never best responses to the opponent's strategies after the second round of deletion.
- We could apply the same procedure again and again after each round of deletion.
- Eventually, we will end up with the intersection:

$$\begin{cases} s_1^* = 1 + b s_2^*; \\ s_2^* = 1 + b s_1^*. \end{cases} \Longrightarrow (s_1^*, s_2^*) = \left(\frac{1}{1-b}, \frac{1}{1-b}\right).$$

Partnership Game: Analysis

Remark. At the intersection, both players play best responses to each other.

Partnership Game: Social Optimum

- Suppose that there is a social planner who could decide how much each agent should work in the partnership.
- Social planner's objective is to maximize total profit net of costs.
- Social planner solves

$$\max_{s_1,s_2} U(s_1, s_2) = \max_{s_1,s_2} 4(s_1 + s_2 + bs_1 s_2) - s_1^2 - s_2^2.$$

Let the optimal effort be s_1^{**} and s_2^{**}.

Partnership Game: Social Optimum

Question. How does s_1^* and s_2^* compare to s_1^{**} and s_2^{**}? Or put it differently, do the agents work too much or too little in the partnership compared to the social optimum?

Partnership Game: Social Optimum

FOCs to the social planner's problem (1.D.3) are

$$\begin{cases} s_1^{**} = 2 + 2bs_2^{**} \\ s_2^{**} = 2 + 2bs_1^{**} \end{cases} \implies (s_1^{**}, s_2^{**}) = (\frac{2}{1-2b}, \frac{2}{1-2b})$$

It is not hard to check that $s_1^{**} > s_1^*$ and $s_2^{**} > s_2^*$.

- Agents work too little compared to the social optimum.
- Intuition: in partnership, at the margin, each agent bear full cost for extra effort she puts in, but benefit is shared with the other agent.

Partnership Game: Social Optimum

Remark. There are three things that we usually do when we face a problem (and also when we write an applied paper):

1. do the mathematical calculation,
2. draw figures,
3. understand the intuition.

1.E. Nash Equilibrium

- We solve the partnership game in Section 1.D.3 by iterated elimination of never best responses.
- In this particular game, we get convergence.
- At the intersection, both players play best responses to each other.
- The property of mutual best responses gives rise to the concept of a Nash Equilibrium, which is formally defined in Definition 1.E.1.

Nash Equilibrium

Definition 1.E.1. A strategy profile $(s_1^*, s_2^*, ..., s_n^*)$ is a Nash equilibrium if, for each player i, s_i^* is a best response to s_{-i}^*:

$$u(s_i^*, s_{-i}^*) \geq u(s_i, s_{-i}^*)$$

for every feasible strategy $s_i \in S_i$; or, s_i^* solves

$$\max_{s_i \in S_i} u(s_i, s_{-i}^*).$$

Nash Equilibrium

Remark (No Regret)**.** At a Nash equilibrium, no player can do strictly better by deviating, holding everyone else's actions fixed.

Remark. A Nash equilibrium can be thought of as self-fulfilling beliefs: Player i would play her Nash strategy if she believes that the other players play their Nash strategies.

Nash Equilibrium

Example 1.E.1. Find Nash equilibrium in the following game.

		P2		
		Left	Center	Right
P1	Up	$(0, \underline{4})$	$(\underline{4}, 2)$	$(5, 3)$
	Middle	$(\underline{4}, 0)$	$(0, \underline{4})$	$(5, 3)$
	Down	$(3, 5)$	$(3, 5)$	$(\underline{6}, \underline{6})$

- We underline the payoffs of P1's (2's) best responses to each of P2's (P1's) strategies.
- NE is where best responses coincide: (Down, Right).

Remark. This game could not be solved using Iterated Elimination of Strictly Dominated Strategies or using Iterated Elimination of Never Best Responses.

For example, rational P1 could choose "Middle" because P1 thinks that P2 would choose "Left". And P1 thinks that P2 would choose "Left" because P2 thinks that P1 would choose "Up". And P1 thinks that P2 thinks that P1 would choose "Up" because P2 thinks that P1 thinks that P2 would choose "Center". And so on.

1.E.1. Relationship between NE and Dominance

Strict Domination

Let us look again at PD Game:

		Prisoner 2	
		C	D
Prisoner 1	C	$(-1, -1)$	$(-9, \underline{0})$
	D	$(\underline{0}, -9)$	$(\underline{-6}, \underline{-6})$

1. For both P1 and P2, D strictly dominates C.
2. Nash equilibrium is (D, D).

Strict Domination

Remark. No strictly dominated strategies would be played in a Nash equilibrium.

- A strictly dominated strategy is not a best response to any strategy of the opponent.
- In particular, it is not a best response to the opponent's strategy in the Nash equilibrium.

Weak Domination

It is possible for a weakly dominated strategy to appear in a Nash equilibrium.

Example 1.E.2. Consider the following game:

		Player 2	
		Left	Right
Player 1	Up	$(\underline{1}, \underline{1})$	$(\underline{0}, 0)$
	Down	$(0, \underline{0})$	$(\underline{0}, \underline{0})$

One of the equilibria (Down, Right) involves the play of weakly dominated strategies.

1.E.2. Coordination Game

Investment Game

- n investors, each could invest either $\$0$ or $\$10$.
- If Investor i invests $\$0$, then she gets $\$0$.
- If Investor i invests $\$10$, then
  - if at least 90% of the investors invest, Investor i gets a profit of $\$15$, or a net profit of $\$5$;
  - if less than 90% of the investors invest, Investor i would lose her initial investment $\$10$.

Investment Game: Analysis

- To look for Nash equilibrium, in principle, we need to look at any possible outcome. For example, 1% of the investors invests and 99% do not.
- There are infinitely many such combinations.
- In practice, we guess and check.
- Guess and check is a very useful method in these games where there are many players, but not many strategies per player.

Investment Game: Analysis

Two Nash equilibria in this game:

1. All investors invest: If all other investors invest, then Investor i's best response is to invest.
2. No investor invests: If all other investors do not invest, then Investor i's best response is not to invest.

Investment Game

Remark. The equilibrium where all investors invest Pareto dominates the equilibrium where no investor invests: every investor is better-off in the first equilibrium.

Remark. Nash equilibrium is a self-fulfilling outcome.

Remark. Unlike the Prisoners' Dilemma game, pre-play communication works in the coordination game.

Remark. This model could help us understand bank runs.

- The good equilibrium: everyone has confidence in the bank and leaves their deposits in the bank. The bank could lend some of the money out on a higher interest rate.
- The bad equilibrium: people lose confidence in the bank and start drawing their deposits out. Then the bank does not have enough cash to cover those deposits and becomes bankrupt.

The Battle of the Sexes

Example 1.E.3.

- Alice and Bob are considering going out for the night.
- While at separate workplaces, Alice and Bob must choose to attend either Opera or Movie without communication.
- Both of them prefer to be together.
- But as for the entertainment, Alice prefers Opera whereas Bob prefers Movie.

The Battle of the Sexes

		Bob: Opera	Bob: Movie
Alice	Opera	$(2, 1)$	$(0, 0)$
	Movie	$(0, 0)$	$(1, 2)$

The Battle of the Sexes

Two (pure strategy) Nash equilibria: (Opera, Opera) and (Movie, Movie).

Remark. Unlike the investment game, there is a conflict of interest between two players in the Battle of the Sexes game.

1.F. Applications

1.F.1. Cournot Model of Duopoly

- Quantities (of a homogeneous product) produced by firms 1 and 2: q_1 and q_2
- Market-clearing price when aggregate quantity is $Q = q_1 + q_2$: $P(Q) = a - Q$.
- Total cost to a firm with q_i: $C_i(q_i) = cq_i$, where $c < a$.
- Firms choose quantities simultaneously.

Normal-Form Representation

- Players: Firm 1 and 2;
- Strategies: $q_i \in S_i = [0, \infty)$ for Firm i;
- Payoffs: For Firm i,

$$\pi_i(q_i, q_j) = q_i[P(q_i + q_j) - c] = q_i[a - (q_i + q_j) - c]$$

The other firm is denoted by j.

Cournot Duopoly

We have learned three ways to solve the problem.

1. Nash equilibrium

2. Best response curves

3. Iterated elimination of never best responses

Nash Equilibrium

(q_1^*, q_2^*) forms a Nash equilibrium if, for each firm i,

$$\pi_i(q_i^*, q_j^*) \geq \pi_i(q_i, q_j^*) \text{ for all feasible } q_i \in S_i.$$

Equivalently, q_i^* solves

$$\max_{q_i \in S_i} \pi_i(q_i, q_j^*) = \max_{q_i \in [0,\infty)} q_i[a - (q_i + q_j^*) - c].$$

FOC yields

$$q_i^* = \frac{1}{2}(a - q_j^* - c).$$

Nash Equilibrium

For (q_1^*, q_2^*) to be a Nash equilibrium, we have

$$\begin{cases} q_1^* = \frac{1}{2}(a - q_2^* - c); \\ q_2^* = \frac{1}{2}(a - q_1^* - c). \end{cases} \implies (q_1^*, q_2^*) = (\frac{a-c}{3}, \frac{a-c}{3}).$$

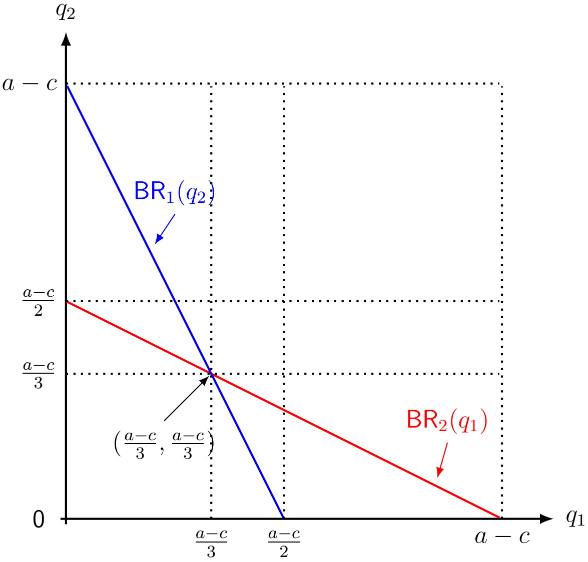
The profit of each firm is

$$\pi_i(q_i^*, q_j^*) = \frac{a-c}{3}\left[a - (\frac{a-c}{3} + \frac{a-c}{3}) - c\right] = \frac{(a-c)^2}{9}$$

Best Response Curves

- We could also solve for the equilibrium graphically using the best response curves.
- The two best response curves $\mathrm{BR}_1(q_2)$ and $\mathrm{BR}_2(q_1)$ intersect once at the equilibrium quantity pair $(q_1^*, q_2^*) = (\frac{a-c}{3}, \frac{a-c}{3})$. (See next page)

Best Response Curves

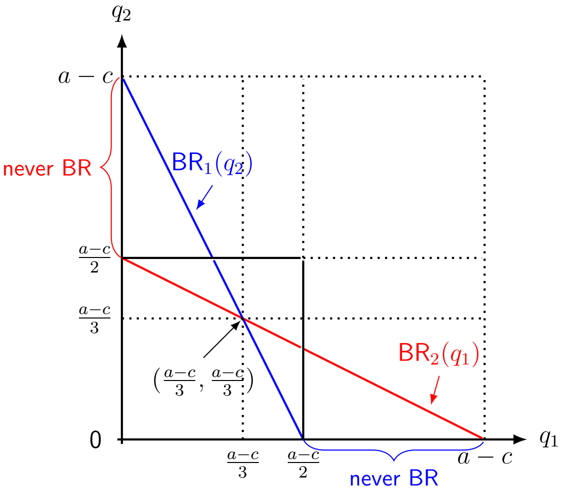


Iterated Elimination of Never Best Responses

- Similar to our analysis of the partnership game, we could apply iterated elimination of never best responses.
- In the first round, we eliminate the quantities higher than the monopoly quantity, i.e., $q_i > q_m = \frac{a-c}{2}$, since $q_i > q_m$ is never a best response against any $q_j \geq 0$.

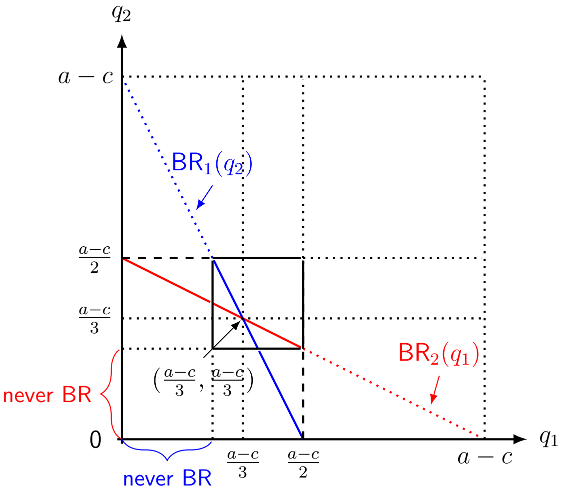
First Round Elimination

$q_i > q_m$ is never a best response.



Second Round Elimination

Given that $q \leq q_m = \frac{a-c}{2}$, $q_i < \frac{a-c}{4}$ is never a best response.



Iterated Elimination of Never Best Responses

Repeating the arguments leads to the equilibrium quantity

$(q_1^*, q_2^*) = (\frac{a-c}{3}, \frac{a-c}{3})$.

Monopoly Case

- Monopolist chooses quantity q to maximize its profit.
- Marginal cost is still c.
- Monopolist's problem is

$$\max_q q(a - q - c).$$

- The solution is

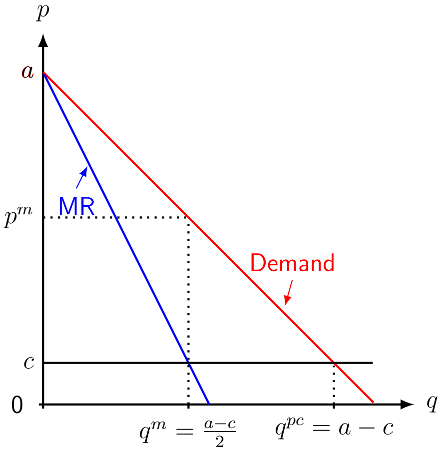
$$q^m = \frac{a-c}{2}.$$

- Total industry profit for the monopoly case is

$$\pi(q^m) = \frac{a-c}{2}\left[a - \left(\frac{a-c}{2}\right) - c\right] = \frac{(a-c)^2}{4}.$$

Monopoly Case

Graphically, intersection of Marginal Revenue (MR) curve and Marginal Cost (MC) curve.



Monopoly Case

- Monopoly quantity $q^m = \frac{a-c}{2}$ is smaller than the total quantity $q_1^* + q_2^* = \frac{2(a-c)}{3}$ produced by Cournot duopoly.
- Monopoly price is higher since $p = a - c - Q$, and total industry profit is higher:

  $\pi(q^m) = \frac{(a-c)^2}{4} > \frac{2(a-c)^2}{9} = \pi_1(q_1^*, q_2^*) + \pi_2(q_1^*, q_2^*)$.

Cournot Duopoly

Question. Each firm in the Cournot duopoly would be better-off sharing the monopoly profit by each producing half of the monopoly quantity. Why don't the firms do that?

Iterated Elimination May not Yield a Unique Solution.

- Consider the three firm version of Cournot Model.
- Let Q_{-i} be the sum of quantities of firms other than i.
- It is still true that any quantity higher than the monopoly quantity $q^m = \frac{a-c}{2}$ is never a best response.
- In the first round of elimination, we eliminate the quantities $q^i > q^m$ for all firms.
- After the first round, we have $Q_{-i} \leq q^m + q^m = a - c$.

Iterated Elimination May Not Yield a Unique Solution

- In this case, any $q_i \in [0, \frac{a-c}{2}]$ is a best response to some $Q_{-i} \in [0, a-c]$.
- Specifically, $\mathsf{BR}_i(Q_{-i}) = \frac{1}{2}(a - Q_{-i} - c)$.
- Therefore, we could not further eliminate any quantities.
- As a result, iterated elimination leads to imprecise prediction in this case.

Remark. The 3-firm Cournot model could still be solved using Nash equilibrium.

1.F.2. Bertrand Model of Duopoly with Differentiated Products

- Prices chosen by firm 1 and 2: p_1 and p_2
- Quantity consumers demand from firm i:

  $q_i(p_i, p_j) = a - p_i + bp_j$, assume $b < 2$.
- Total cost: $C_i(q_i) = cq_i$ where $c < a$.
- Firms act simultaneously.

Normal-Form Representation

- Players: Firm 1 and 2;
- Strategies: $p_i \in S_i = [0, \infty)$ for Firm i;
- Payoffs: For Firm i:

$$\pi_i(p_i, p_j) = q_i(p_i, p_j)(p_i - c) = (a - p_i + bp_j)(p_i - c).$$

Nash Equilibrium

(p_i^*, p_j^*) forms a Nash equilibrium if, for each firm i, q_i^* solves

$$\max_{p_i \in [0,\infty)} \pi_i(p_i, p_j^*) = \max_{p_i \in [0,\infty)} (a - p_i + bp_j^*)(p_i - c).$$

The solution to this optimization problem is

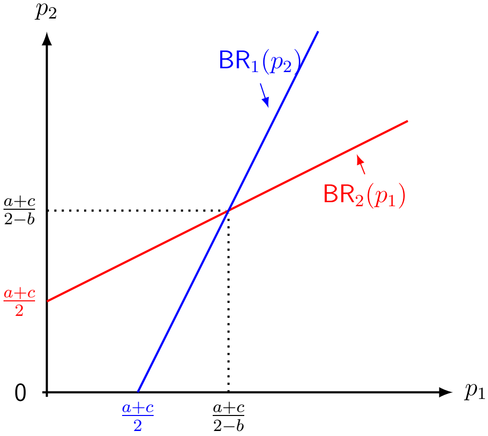
$$p_i^* = \frac{1}{2}(a + bp_j^* + c).$$

For (p_1^*, p_2^*) to be a Nash equilibrium, we have

$$\begin{cases} p_1^* = \frac{1}{2}(a + bp_2^* + c); \\ p_2^* = \frac{1}{2}(a + bp_1^* + c). \end{cases} \implies (p_1^*, p_2^*) = \Big(\frac{a+c}{2-b}, \frac{a+c}{2-b}\Big).$$

Best Response Curves

The two best response curves $\text{BR}_1(p_2)$ and $\text{BR}_2(p_1)$ intersect once at the equilibrium quantity pair $(p_1^*, p_2^*) = (\frac{a+c}{2-b}, \frac{a+c}{2-b})$.



1.F.3. Candidate-Voter Model

This model is a simplified version of Osborne and Slivinski (1996).

- There are n voters, with positions $1, ..., n$.
- Voters vote for the closest candidate.
- Unlike the previous voting model in Section 1.C.3:

    1. The number of candidates is not fixed.
    2. Candidates cannot choose their position. Each voter is a potential candidate.

Normal-Form Representation

- Players: Voters/Candidates;
- Strategies: to run or not to run;
- Payoffs:
  - Prize if win $= B$, $1/k$ chances of winning if k candidates win;
  - Cost of running $= c$, assuming $B \geq 2c$;
  - If Voter/Candidate i at position x and the winner is at y, then i gets $-|x-y|$.

Candidate-Voter Model: An example

Suppose that there are 11 voters.

- If Voter/Candidate at position 5 runs and is the sole winner, then she gets $B - c$;
- If Voter/Candidate at position 5 runs and Voter/Candidate at position 7 wins, then Voter/Candidate at position 5 gets $-c - 2$;
- If Voter/Candidate at position 5 does not run and Voter/Candidate at position 7 wins, then Voter/Candidate at position 5 gets -2.

Nash Equilibrium

Question. Is there any NE where no candidate runs?

Nash Equilibrium

No. If no candidate runs, every possible candidate would be better-off running: 0 if not running v.s. $B - c$ if running.

Nash Equilibrium

Question. Is there any NE where one candidate runs?

Nash Equilibrium

Yes if n is odd.

Only the center candidate running constitutes a NE:

- For center candidate, when no other candidate runs, her BR is running: 0 if not running v.s. $B - c$ if running.
- For other candidates, their BR are not running since they would lose if running and running costs c.

Nash Equilibrium

Question. Is there any NE where two candidates run?

Nash Equilibrium

Yes. The two candidates needs to be of equal distance from the center and cannot be too far apart.

- For example, in the 11-Voter/Candidate case, Voter/Candidate 5 and 7 running is a NE.
- We need to check the following three types of deviations:
  - A Voter/Candidate from the outside enters (position $1-4$ and $8-11$).
  - The Voter/Candidate in the middle enters (position 6).
  - Voter/Candidate 5 and 7 choose not to run.

Nash Equilibrium

Remark. There are many NE in this Voter/Candidate Model.

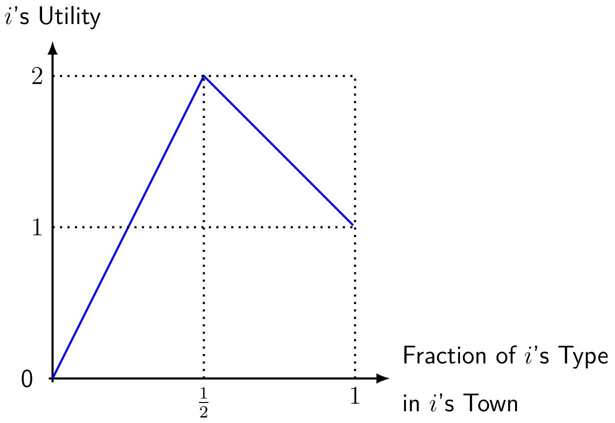
And not all of the NE predict candidates “at the center”.

1.F.4. Location Model

- There are two types of people in the society, namely, Tall (T) and Short (S).
- The measure of T and S are both 1.
- There are two towns, namely, East (E) and West (W).
- Each town could hold measure 1 of people.
- All people simultaneously choose which town to live in.
- If more than measure 1 of people chooses one town, then we randomly choose who could stay.

Location Model

The payoff of everyone is the same.



Nash Equilibrium

There are three Nash Equilibria as follows (Guess and Check):

1. Two Segregated Equilibria

   a) All T in E and All S in W;

   b) All T in W and All S in E;

2. One Integrated Equilibrium: exactly $\frac{1}{2}$ of T and $\frac{1}{2}$ of S in one town.

Remark. The two segregated equilibria are stable whereas the integrated equilibrium is unstable.

- If starting from $99\%/1\%$ (a small deviation away from the segregated equilibrium), the population will eventually restore to the segregated equilibrium;
- If starting from $51\%/49\%$ (a small deviation away from the integrated equilibrium), the population will eventually become segregated.

Location Model

Remark. Observing segregation does not imply people's preference for segregation. In this location model, everyone prefers integration: an individual obtains 2 in the integrated equilibrium whereas in the segregated equilibrium, she only gets 1. However, segregation equilibria exist and are stable. This idea is brought up by Schelling (1971).

Location Model

Remark. Actually, according to the description of the game, everyone choosing the same town and getting randomized is also an equilibrium. Integration is attained.

Remark. The integration outcome could also be obtained via individual randomization. This idea would be clearer after we formally discuss Mixed Strategy Nash Equilibrium in the next section.

1.G. Mixed Strategy Nash Equilibrium

Example 1.G.1. Find Nash equilibrium in the Rock, Paper, Scissors game.

Player 2

		Rock	Paper	Scissors
	Rock	$(0,0)$	$(-1,\underline{1})$	$(\underline{1},-1)$
Player 1	Paper	$(\underline{1},-1)$	$(0,0)$	$(-1,\underline{1})$
	Scissors	$(-1,\underline{1})$	$(\underline{1},-1)$	$(0,0)$

Applying previous definition of Nash Equilibrium, Definition 1.E.1, there exists no (Pure Strategy) NE.

Rock, Paper, Scissors

Question. How do you play the Rock, Paper, Scissors game?

Mixed Strategy Nash Equilibrium

Definition 1.G.1 (Mixed Strategy)**.** Suppose that Player i has K pure strategies: $S_i = \{s_{i1}, ..., s_{iK}\}$. Then a mixed strategy for Player i is a probability distribution $p_i = (p_{i1}, ..., p_{iK})$ over S_i, where $0 \leq p_{ik} \leq 1$ for $k = 1, ..., K$ and $p_{i1} + ... + p_{iK} = 1$.

Remark. When $p_{ij} = 1$ and $p_{ik} = 0$ for all $k \neq j$, the mixed strategy $p_i = (p_{i1}, ..., p_{iK})$ is the pure strategy s_{ij}.

Mixed Strategy Nash Equilibrium

Definition 1.G.2. A mixed strategy profile $(p_1^*, p_2^*, ..., p_n^*)$ is a mixed strategy Nash equilibrium if, for each player i, p_i^* is a best response to p_{-i}^*.

Rock, Paper, Scissors

- Formally, your mixed strategy for Rock, Paper, Scissors game is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
- We show that the strategy profile $((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$ is a NE by showing that $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a best response to the opponent's strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
- Need to check: against $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, expected payoff from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is (weakly) higher than expected payoff from any other mix $(p, q, 1 - p - q)$.

Matching Pennies

Question. Could you check that $\left((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\right)$ is a (mixed strategy) Nash equilibrium for the Matching Pennies game?

Player 2

Player 1		Head	Tail
	Head	$(-1, \underline{1})$	$(\underline{1}, -1)$
	Tail	$(\underline{1}, -1)$	$(-1, \underline{1})$

1.G.1. Finding Mixed Strategy Nash Equilibrium

Example 1.G.2. Let us revisit the Battle of the Sexes game:

		Bob	
		Opera	Movie
Alice	Opera	$(2,1)$	$(0,0)$
	Movie	$(0,0)$	$(1,2)$

The Battle of the Sexes

- Given Alice's and Bob's strategies, we could calculate their payoffs.
- For example, consider Alice's mixed strategy $p_A = (\frac{1}{5}, \frac{4}{5})$ and Bob's mixed strategy $p_B = (\frac{1}{2}, \frac{1}{2})$.
- Then Alice's expected payoffs from Opera and Movie are respectively

$$\mathbb{E}U_A(\textsf{Opera}, p_B) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1;$$
$$\mathbb{E}U_A(\textsf{Movie}, p_B) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

The Battle of the Sexes

- Alice's expected payoff from mixed strategy $p = (\frac{1}{5}, \frac{4}{5})$ is

$$\begin{aligned} \mathbb{E}U_A(p_A, p_B) =& \frac{1}{5} \cdot \mathbb{E}U_A(\mathsf{Opera}, p_B) + \frac{4}{5} \cdot \mathbb{E}U_A(\mathsf{Movie}, p_B) \\ =& \frac{1}{5} \cdot 1 + \frac{4}{5} \cdot \frac{1}{2} = \frac{3}{5}. \end{aligned}$$

The Battle of the Sexes

Observation. Alice's expected payoff from the mixed strategy p_A is the weighted average of the expected payoffs from each of the pure strategies in the mix. And further, the weighted average always lies in-between the lowest and the highest payoffs involved in the mix.

- In our example, $\frac{3}{5} = \frac{1}{5} \cdot 1 + \frac{4}{5} \cdot \frac{1}{2} \in [\frac{1}{2}, 1]$.
- This observation is true in general.

Finding Mixed Strategy Nash Equilibrium

Result. If a mixed strategy is a best response, then each pure strategy in the mix must be best responses. In particular, each must yield the same expected payoff.

The Battle of the Sexes

Applying this idea, we would be able to find the mixed strategy Nash equilibrium for the Battle of the Sexes game.

- Assume $p_A = (p, 1-p)$ and $p_B = (q, 1-q)$.
- Alice's expected payoffs from Opera and Movie are

$$\mathbb{E}U_A(\text{Opera}, p_B) = q \cdot 2 + (1-q) \cdot 0 = 2q;$$

$$\mathbb{E}U_A(\text{Movie}, p_B) = q \cdot 0 + (1-q) \cdot 1 = 1-q.$$

- For the mixed strategy to be a best response,

$$\mathbb{E}U_A(\text{Opera}, p_B) = \mathbb{E}U_A(\text{Movie}, p_B) \implies q = \frac{1}{3}.$$

The Battle of the Sexes

- Similarly, for Bob

$$\begin{cases} \mathbb{E}U_B(p_A, \textsf{Opera}) = p \cdot 1 + (1-p) \cdot 0 = p \\ \mathbb{E}U_B(p_A, \textsf{Movie}) = p \cdot 0 + (1-p) \cdot 2 = 2 - 2p \end{cases}$$

$$\mathbb{E}U_B(p_A, \textsf{Opera}) = \mathbb{E}U_B(p_A, \textsf{Movie}) \implies p = \frac{2}{3}.$$

- The mixed strategy Nash equilibrium is

$$\left(p_A = (\frac{2}{3}, \frac{1}{3}), p_B = (\frac{1}{3}, \frac{2}{3})\right).$$

The Battle of the Sexes

Remark. Notice that using Alice's payoff and applying the indifferent condition, we solve for Bob's mixing, i.e., $p_B = (q, 1-q)$, and similarly, using Bob's payoff, we solve for Alice's mixing, i.e., $p_A = (p, 1-p)$.

Remark. A maybe easier to remember version: Bob's equilibrium mix makes Alice indifferent and Alice's equilibrium mix make Bob indifferent.

Mixed Strategy Nash Equilibrium

- To check whether the mixed strategy is indeed a Nash equilibrium, previously for the Rock, Paper, Scissors game, we checked that there is no strictly profitable deviation to all pure strategies and all other possible mixed strategies.
- Actually, checking all pure strategies are sufficient.

Question. Why is it that checking all pure strategies are sufficient?

1.G.2. Other Examples

Tax Paying Game

		Taxpayer	
		Honest	Cheat
Auditor	Audit	$(2, 0)$	$(4, -10)$
	Not Audit	$(4, 0)$	$(0, 4)$

Tax Paying Game

Let us first look for pure strategy Nash equilibrium.

		Taxpayer	
		Honest (H)	Cheat (C)
Auditor	Audit (A)	$(2, \underline{0})$	$(\underline{4}, -10)$
	Not Audit (N)	$(\underline{4}, 0)$	$(0, \underline{4})$

There is no pure strategy Nash equilibrium in this game.

Tax Paying Game

Next, we look for mixed strategy Nash equilibrium.

- Let the auditor's mixed strategy be $p_A = (p, 1-p)$ and the taxpayer's mixed strategy be $p_T = (q, 1-q)$.
- We use the auditor's payoff to find $(q, 1-q) = (\frac{2}{3}, \frac{1}{3})$.
- We use the taxpayer's payoff to find $(p, 1-p) = (\frac{2}{7}, \frac{5}{7})$.

Tax Paying Game

Question. What if we raise the fine to -20. Will such a policy raise tax compliance rate q?

Question. What policies would raise tax compliance rate q?

Penalty Kick Game

		Goalkeeper	
		Left (L)	Right (R)
	left (l)	$(4, -4)$	$(9, -9)$
Shooter	middle (m)	$(6, -6)$	$(6, -6)$
	right (r)	$(9, -9)$	$(4, -4)$

The mixed strategy Nash equilibrium is

$$\left(p_S = (\frac{1}{2}, 0, \frac{1}{2}), p_G = (\frac{1}{2}, \frac{1}{2})\right).$$

Penalty Kick Game

Remark. Recall that when we first study the Penalty Kick Game in Section 1.D.2, we have shown that m is never a best response to any belief. Therefore, it should not be surprising that in the mixed strategy equilibrium, m is not played.

1.G.3. Dominance and Best Responses

- Recall that if s_i is strictly dominated, then there is no belief that Player i could hold (about the other players' strategies) such that it would be optimal to play s_i.
- The converse is also true for 2-player case, provided we allow for mixed strategies.

Dominance and Best Responses

We will illustrate the second result in an example.

- Consider the Penalty Kick Game.
- In the Penalty Kick Game, we know that m is never a best response for Shooter.
- Now we show that there exists a (mixed) strategy that strictly dominates m. Let such a strategy be $(p, 0, 1-p)$.

Dominance and Best Responses

Playing against Goalkeeper's strategy $(q, 1-q)$ for $q \in [0,1]$,

1. m gives 6;
2. the mixed strategy $(p, 0, 1-p)$ gives $4pq + 9p(1-q) + 9(1-p)q + 4(1-p)(1-q) = (-10p+5)q + 5p + 4$.
3. For the mixed strategy to dominate m for any $q \in [0,1]$, we need

   - $5p + 4 > 6$ (when $q = 0$) and
   - $-10p + 5p + 9 > 6$ (when $q = 1$).

   Therefore, any mix with $p \in (\frac{2}{5}, \frac{3}{5})$ will do.

1.G.4. Interpretations of Mixed Strategies

1. People literally randomizing: e.g., players in rock, paper and scissors game and in penalty kick game.
2. Beliefs of others' actions: e.g., in the battle of the sexes game, we could think about Alice's mixture as what Bob believes that Alice is going to do. Holding such a belief, Bob is indifferent between the two actions.
3. Proportion of players: e.g., taxpayer's strategy $(\frac{2}{3}, \frac{1}{3})$ in the tax paying game can be thought of as the proportion of taxpayers being honest $(\frac{2}{3})$ and cheating $(\frac{1}{3})$ on taxes.