# Convex sets and (quasi)-concave/convex functions

## 1.A. Convex Sets

**Definition 1.A.1** (Convex Set)**.** A set $\mathcal{S}$ of points in $n$-dimensional space is called *convex* if, given any two points $x^a = (x_1^a, x_2^a, ..., x_n^a)$ and $x^b = (x_1^b, x_2^b, ..., x_n^b)$ in $\mathcal{S}$ and any real number $\alpha \in [0, 1]$, the point $\alpha x^a + (1-\alpha)x^b = (\alpha x_1^a + (1-\alpha)x_1^b, ..., \alpha x_n^a + (1-\alpha)x_n^b)$ is also in $\mathcal{S}$.

A geometric test of convexity is that given any two points of the set, the whole line segment joining them should lie in the set.

Figure 1 and 2 are examples of *convex* sets. Please be aware that to apply the geometric test of convexity, we need to ensure that for *any* two points of the set, the whole line segment lie in the set.

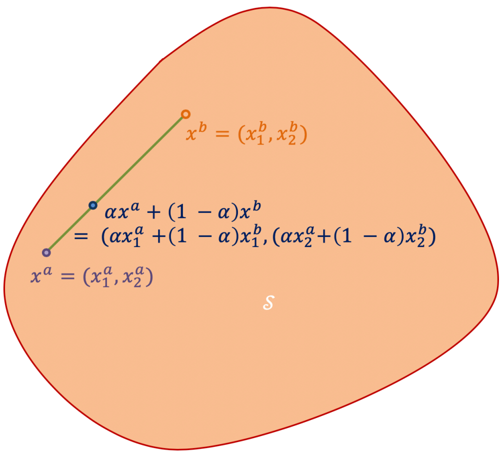


Figure 1: Convex Set (a)

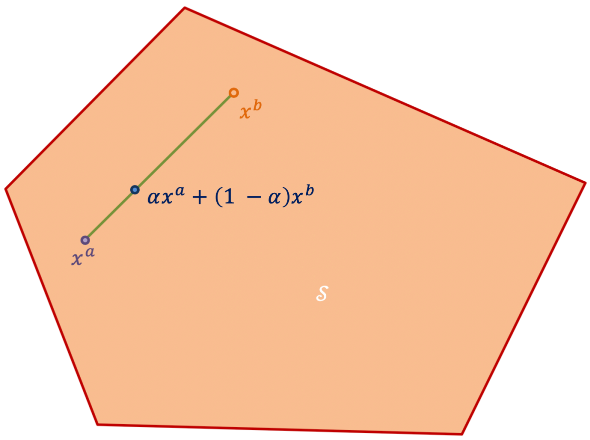


Figure 2: Convex Set (b)

Figure 3 and 4 are examples of *non-convex* sets. The sets are *non-convex*, since there exist points $x^a$ and $x^b$ and a real number $\alpha$, such that the point $\alpha x^a + (1-\alpha)x^b$ is not inside the set.

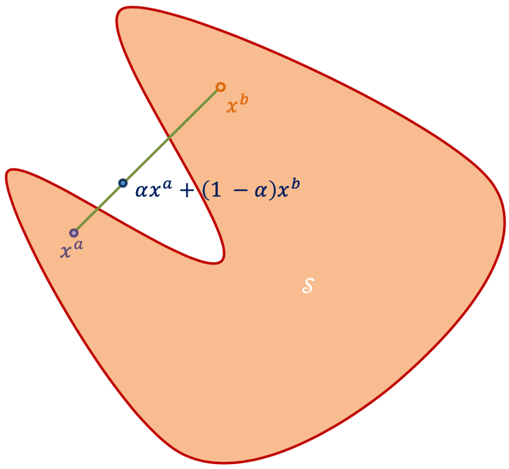


Figure 3: Non-Convex Set (a)

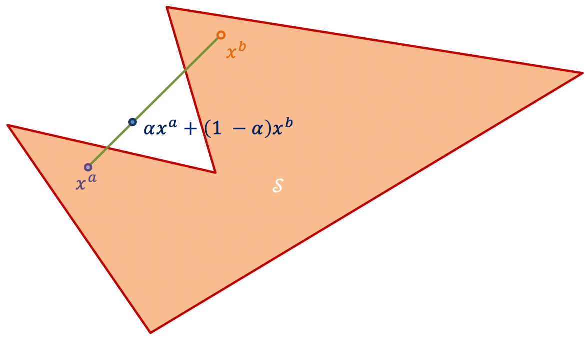


Figure 4: Non-Convex Set (b)

## 1.B. Quasi-convex/concave functions

**Definition 1.B.1** (Quasi-convex Function)**.** A function $f : \mathcal{S} \to \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, is *quasi-convex* if the set $\{x|f(x) \leq c\}$ is convex for all $c \in \mathbb{R}$, or equivalently, if

$$f(\alpha x^a + (1-\alpha)x^b) \leq \max\{f(x^a), f(x^b)\}, \tag{1}$$

for all $x^a$, $x^b$ and for all $\alpha \in [0,1]$.

We show the equivalence of

(a) The set $\{x|f(x) \leq c\}$ is convex for all $c \in \mathbb{R}$;

(b) $f(\alpha x^a + (1-\alpha)x^b) \leq \max\{f(x^a), f(x^b)\}$, for all $x^a$, $x^b$ and for all $\alpha \in [0,1]$.

**Proof.** (a) $\implies$ (b): Since (a) holds for all $c \in \mathbb{R}$, for any $x^a$ and $x^b$, we could set $c = \max\{f(x^a), f(x^b)\}$. Then since $f(x^a) \leq \max\{f(x^a), f(x^b)\} = c$, $f(x^b) \leq \max\{f(x^a), f(x^b)\} = c$, by (a), we have $f(\alpha x^a + (1-\alpha)x^b) \leq c = \max\{f(x^a), f(x^b)\}$ for any $\alpha \in [0,1]$. Thus, (b) holds.

(b) $\implies$ (a): Equivalently, we show "not (a) $\implies$ not (b)".

If (a) fails, then there exists $x^a$, $x^b$, $c$ and $\alpha \in [0,1]$ such that $f(x^a) \leq c$ and $f(x^b) \leq c$ but $f(\alpha x^a + (1-\alpha)x^b) > c$. Then $f(\alpha x^a + (1-\alpha)x^b) > c \geq \max\{f(x^a), f(x^b)\}$. Thus, (b) fails for these values of $x^a$, $x^b$ and $\alpha$. □

The definition of *quasi-concave* is given in Definition 1.B.2 below.

**Definition 1.B.2** (Quasi-concave Function)**.** A function $f : \mathcal{S} \to \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, *quasi-concave* if the set $\{x|f(x) \geq c\}$ is convex for all $c \in \mathbb{R}$, or equivalently, if $f(\alpha x^a + (1-\alpha)x^b) \geq \min\{f(x^a), f(x^b)\}$, for all $x^a$, $x^b$ and for all $\alpha \in [0, 1]$.

## 1.C. *Quasi-convexity* (*quasi-concavity*) and *convexity* (*concavity*)

The *quasi* in Definition 1.B.1 and 1.B.2 serves to distringuish them from stronger properties of *convexity* and *concavity*. Formally, we define convexity as follows.

**Definition 1.C.1** (Convex Function)**.** A function $f : \mathcal{S} \to \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, is *convex* if

$$f(\alpha x^a + (1-\alpha)x^b) \leq \alpha f(x^a) + (1-\alpha)f(x^b), \tag{2}$$

for all $x^a$, $x^b$ and for all $\alpha \in [0, 1]$.

(2) *convexity* implies (1) *quasi-convexity* since

$$\begin{aligned} f(\alpha x^a + (1-\alpha)x^b) &\underbrace{\leq}_{(2)} \alpha f(x^a) + (1-\alpha)f(x^b) \\ &\leq \alpha \max\{f(x^a), f(x^b)\} + (1-\alpha)\max\{f(x^a), f(x^b)\} \\ &= \max\{f(x^a), f(x^b)\}. \end{aligned}$$

In other words, a convex function must be quasi-convex.

Similarly, we could define concavity and compare it with *quasi-concavity*.

**Definition 1.C.2** (Concave Function)**.** A function $f : \mathcal{S} \to \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, is *concave* if

$$f(\alpha x^a + (1-\alpha)x^b) \geq \alpha f(x^a) + (1-\alpha)f(x^b), \tag{3}$$

for all $x^a$, $x^b$ and for all $\alpha \in [0, 1]$.

Following the same logic, we could show that a concave function must be quasi-concave. Figure 5 provides a graphical illustration of a *concave* function for the one-variable case. The red dot (LHS of (3)) is always higher than the green dot (RHS of (3)).

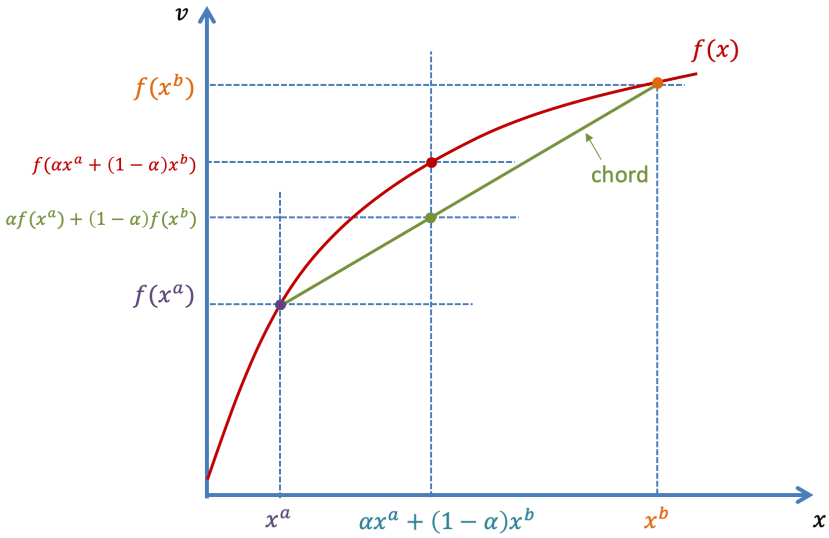


Figure 5: Concave Function

The graph of the function lies on or above the chord joining any two points of it.

## 1.D. More on concave functions

An alternative interpretation of a concave function is sometimes useful. Consider the $(n+1)$-dimensional space consisting of points like $(x, v)$ where $x$ is an $n$-dimensional vector and $v$ is a scalar. Define the set $\mathcal{F} = \{(x, v) | v \leq f(x)\}$.

Then, we make the following claim:

**Claim.** $f$ is a concave function if and only if $\mathcal{F}$ is a convex set.

**Proof.** "$\Longrightarrow$": To prove that $\mathcal{F}$ is a convex set, we need to show that for all $(x^a, v^a)$ and $(x^b, v^b)$ that satisfy $v^a \leq f(x^a)$ and $v^b \leq f(x^b)$ and any real number $\alpha \in [0, 1]$, we have $\alpha v^a + (1-\alpha)v^b \leq f(\alpha x^a + (1-\alpha)x^b)$.

By concavity of $f$, we know that for all $x^a$ and $x^b$ and for all $\alpha \in [0, 1]$, (3) holds.

Therefore, for all $(x^a, v^a)$ and $(x^b, v^b)$ that satisfy $v^a \leq f(x^a)$ and $v^b \leq f(x^b)$ and any real number $\alpha \in [0, 1]$,

$$\alpha v^a + (1-\alpha)v^b - \left[f(\alpha x^a + (1-\alpha)x^b)\right]$$

$$\underbrace{\leq}_{(3)} \alpha v^a + (1-\alpha)v^b - \left[\alpha f(x^a) + (1-\alpha)f(x^b)\right]$$

$$\underbrace{\leq}_{v^a \leq f(x^a) \text{ and } v^b \leq f(x^b)} \alpha v^a + (1-\alpha)v^b - \left[\alpha v^a + (1-\alpha)v^b\right] = 0$$

Therefore, $\alpha v^a + (1-\alpha)v^b \leq f(\alpha x^a + (1-\alpha)x^b)$ and convexity of set $\mathcal{F}$ follows.

"$\Longleftarrow$": To prove that $F$ is concave, we need to show that for all $x^a$, $x^b$ and all $\alpha \in [0,1]$, (3) holds.

For any $x^a$ and $x^b$, set $v^a = f(x^a)$ and $v^b = f(x^b)$, so that $v^a \leq f(x^a)$ and $v^b \leq f(x^b)$ are satisfied, i.e., $(x^a, v^a) \in \mathcal{F}$ and $(x^b, v^b) \in \mathcal{F}$. Then by convexity of set $\mathcal{F}$, for any real number $\alpha \in [0,1]$, we have $\alpha v^a + (1-\alpha)v^b \leq f(\alpha x^a + (1-\alpha)x^b) \underbrace{\Longrightarrow}_{v^a=f(x^a),\, v^b=f(x^b)} \alpha f(x^a) + (1-\alpha)f(x^b)) \leq f(\alpha x^a + (1-\alpha)x^b)$, and concavity of the function $f$ follows. □

The claim could be more easily understood graphically. Figure 6 illustrates the case with a scalar variable $x$. The function $f$ is the red curve. The set $\mathcal{F}$ is the area shaded in orange. The claim means that the concave function $f$ traps a convex set $\mathcal{F}$ underneath its graph. And it is clear from Figure 6.

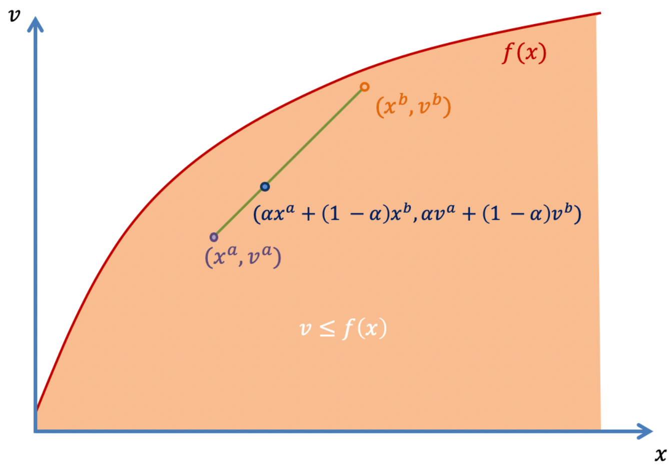


Figure 6: Concave Function

For differentiable functions, the concavity property could be interpreted in terms of first-order derivatives. We have also shown a similar graph to Figure 7 below, and interpreted *concavity* graphically: the graph of the function lies on or above the chord joining any two points of it. To express the concavity of $f(x)$ in terms of its derivative, we now draw the tangent to $f(x)$ at $x^a$. The requirement of concavity says that the graph of the function should lie on or below the tangent. Or expressed differently,

$$f_x(x^a)(x^b - x^a) \geq f(x^b) - f(x^a),$$

where $f_x(x^a)$ is the slope of the tangent to $f(x)$ at $x^a$.

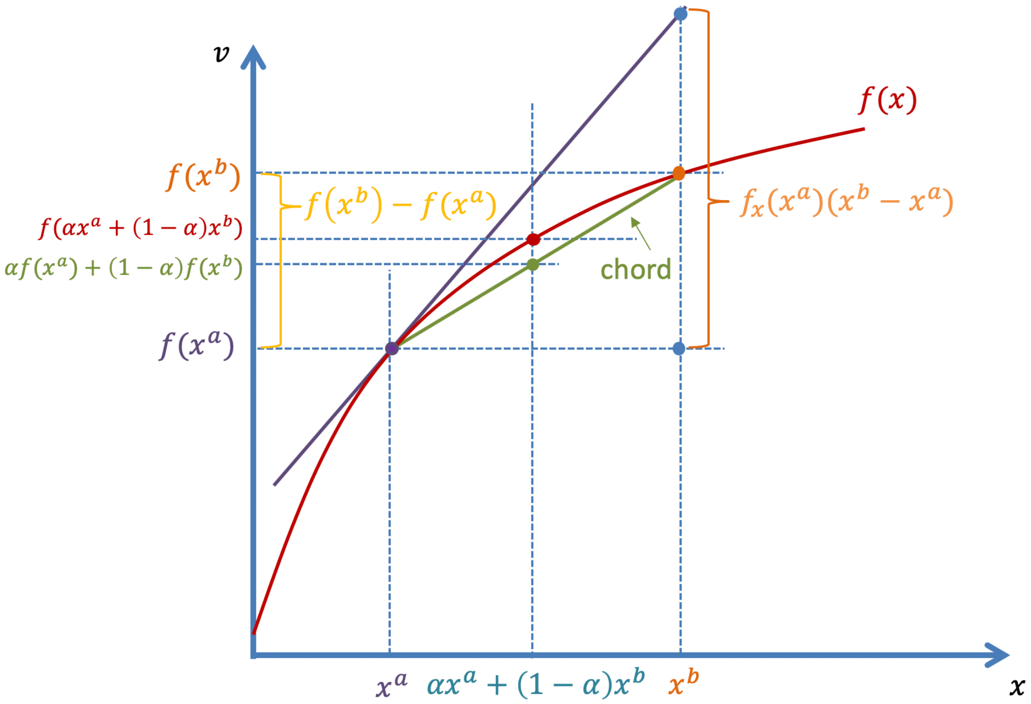


Figure 7: Concave Function

Such an expression holds for higher dimensions. The result is summarized in Proposition 1.D.1 below.

**Proposition 1.D.1** (Concave Function)**.** *A differentiable function* $f : \mathcal{S} \to \mathbb{R}$*, defined on a convex set* $\mathcal{S} \subset \mathbb{R}^N$*, is concave if and only if*

$$f_x(x^a)(x^b - x^a) \geq f(x^b) - f(x^a), \tag{7.1}$$

*for all* $x^a, x^b \in \mathcal{S}$ .

For twice continusouly differentiable functions, this concavity property could be interpreted in terms of second-order derivatives.

**Proposition 1.D.2.** *The (twice continuously differentiable) function* $f : \mathcal{S} \to \mathbb{R}$ *is concave if and only if* $f_{xx}$ *is negative semi-definite for every* $x \in \mathcal{S}$*. If* $f_{xx}$ *is negative definite for every* $x \in \mathcal{S}$*, then the function is strictly concave.*