Game Theory
Assignment 1 Solution

注：此答案步骤较简略，仅供参考。

Question 1: Dominated Strategies and Nash Equilibrium Consider the following game:

		Player 2		
		Left (L)	Center (C)	Right (R)
Player 1	Up (U)	$(2,3)$	$(1,1)$	$(2,1)$
	Middle (M)	$(1,2)$	$(2,1)$	$(3,3)$
	Down (D)	$(0,1)$	$(3,0)$	$(1,1)$

(a) For each player, are there any strictly dominated strategies? If yes, state them.

(b) State your prediction of the outcome using *Iterated Elimination of Strictly Dominated Strategies*.

(c) Find all pure strategy Nash equilibria.

(d) Find all mixed strategy Nash equilibria. (Hint: You may utilize the result in (b))

Solution:

(a) Player 2's strategy C is strictly dominated by strategy L.

(b) IESDS outcome is

		Player 2	
		Left (L)	Right (R)
Player 1	Up (U)	$(2,3)$	$(2,1)$
	Middle (M)	$(1,2)$	$(3,3)$

(c) (U, L) and (M, R)

(d) $\left((\frac{1}{3}, \frac{2}{3}, 0), (\frac{1}{2}, 0, \frac{1}{2})\right)$

Question 2: Cournot Model

(i) (N **firms**) There are N firms in the market. Let q_1, q_2,...,q_N denote the quantities (of a homogeneous product) produced by the N firms respectively. Let $P(Q) = a - Q$ be the market-clearing price when the aggregate quantity on the market is $Q = \sum_{n=1}^{N} q_n$. Assume that the total cost to a firm with quantity q_i is $C_i(q_i) = cq_i$, where $c < a$. Following Cournot, suppose that the firms choose their quantities simultaneously. What is the Nash equilibrium of the game?

(ii) (**heterogeneous costs**) There are 2 firms. Let q_1 and q_2 denote the quantities (of a homogeneous product) produced by firms 1 and 2, respectively. Let $P(Q) = a - Q$ be the market-clearing price when the aggregate quantity on the market is $Q = q_1 + q_2$. Assume that the total cost to a firm with quantity q_i is $C_i(q_i) = c_i q_i$, where $c_1 < c_2$. Following Cournot, suppose that the firms choose their quantities simultaneously.

(a) What is the Nash equilibrium if $0 < c_i < a/2$ for each firm?

(b) What if $c_1 < c_2 < a$ but $2c_2 > a + c_1$?

Solution

(i) Suppose that $(q_1^*, q_2^*, ..., q_N^*)$ constitutes a pure-strategy Nash equilibrium. Then q_i^* solves

$$\max_{q_i}(a - \sum_{j \neq i} q_j^* - q_i - c)q_i.$$

First order condition implies

$$q_i^* = \frac{a - c - \sum_{j \neq i} q_j^*}{2}$$

By symmetry, $q_1^* = q_2^* = ... = q_N^*$ and the solution is

$$q_i^* = \frac{a - c}{N + 1} \text{ for all } i = 1, ..., N.$$

(ii) (a) Suppose that (q_1^*, q_2^*) constitutes a pure-strategy Nash equilibrium.

If $q_1^* > 0$, $q_2^* > 0$, then solving from the first-order conditions:

$$q_1^* = \frac{a - 2c_1 + c_2}{3};$$
$$q_2^* = \frac{a - 2c_2 + c_1}{3}.$$

This is the case when $0 < c_i < a/2$.

(b) If $2c_2 > a + c_1$, then q_2^* solved in part (a) is negative. Thus, we conjecture $q_2^* = 0$. Firm 1's best response is

$$q_1^* = \frac{a - c_1}{2}.$$

As a final step, we compute firm 2's profit function given q_1^* and show that $q_2^* = 0$ is indeed a best response to $q_1^* = \frac{a-c_1}{2}$.

$$\pi_2 = (a - q_1^* - q_2 - c_2)q_2 = (\frac{a + c_1 - 2c_2}{2} - q_2)q_2;$$
$$\frac{\partial \pi_2}{\partial q_2} = \frac{1}{2}(a + c_1 - 2c_2) - 2q_2.$$

Since $\partial\pi_2/\partial q_2 < 0$ for any $q_2 \geq 0$, it is optimal for firm 2 to choose $q_2^* = 0$.

Hence, $((a - c_1)/2, 0)$ constitutes a Nash equilibrium.

Question 3: Gibbons 1.13 Each of two firms has one job opening. Suppose that (for reasons not discussed here but relating to the value of filling each opening) the firms offer different wages: firm i offers the wage w_i, where $(1/2)w_1 < w_2 < 2w_1$. Imagine that there are two workers, each of whom can apply to only one firm. The workers simultaneously decide whether to apply to firm 1 or firm 2. If only one worker applies to a given firm, that worker gets the job; if both workers apply to one firm, the firm hires one worker at random and the other worker is unemployed (which has a payoff of zero). Solve for the Nash equilibria of the workers' normal-form game.

		Worker 2	
		Apply to Firm 1	Apply to Firm 2
Worker 1	Apply to Firm 1	$(\frac{1}{2}w_1, \frac{1}{2}w_1)$	(w_1, w_2)
	Apply to Firm 2	(w_2, w_1)	$(\frac{1}{2}w_2, \frac{1}{2}w_2)$

Solution

1. Two pure-strategy Nash equilibria: (Apply to Firm 2, Apply to Firm 1) and (Apply to Firm 1, Apply to Firm 2).

2. We solve for the mixed strategy Nash equilibrium. Let $(q, 1-q)$ denote worker 1's mixed strategy in which he/she plays "Apply to Firm 1" with probability q, and $(r, 1-r)$ denote worker 2's mixed strategy in which he/she plays "Apply to Firm 1" with probability r.

   If $((q^*, 1-q^*), (r^*, 1-r^*))$ constitutes a mixed-strategy Nash equilibrium, then given worker 2's strategy, worker 1's expected payoff from choosing "Apply to Firm 1" and "Apply to Firm 2" should be equal. That is,

$$r^* \cdot \frac{1}{2}\omega_1 + (1-r^*) \cdot \omega_1 = r^* \cdot \omega_2 + (1-r^*) \cdot \frac{1}{2}\omega_2. \tag{1}$$

   Similarly,

$$q^* \cdot \frac{1}{2}\omega_1 + (1-q^*) \cdot \omega_1 = q^* \cdot \omega_2 + (1-q^*) \cdot \frac{1}{2}\omega_2. \tag{2}$$

   Combine (1) and (2), we have:

$$q^* = r^* = \frac{2\omega_1 - \omega_2}{\omega_1 + \omega_2}.$$

Question 4: Hotelling's Location Game (Polak PS1) Recall the voting game we discussed in class. There are two candidates, each of whom chooses a position from the set $S_i := \{1, 2, ..., 10\}$. The voters are equally distributed across these ten positions. Voters vote for the candidate whose position is closest to theirs. If the two candidates are equidistant from a given position, the voters at that position split their votes equally. The aim of the candidates is to maximize their percentage of the total vote. Thus, for example, $u_1(8,8) = 50\%$ and $u_1(7,8) = 70\%$.

(i) In class, we showed that strategy 2 strictly dominates strategy 1. In fact, other strategies strictly dominate strategy 1. Find all the strategies that strictly dominate strategy 1. Explain your answer.

(ii) Suppose now that there are three candidates. Thus, for example, $u_1(8,8,8) = 33.\dot{3}\%$ and $u_1(7,9,9) = 73.\dot{3}\%$.

(a) Is strategy 1 dominated, strictly or weakly, by strategy 2? Explain.

(b) Is strategy 1 dominated, strictly or weakly, by strategy 3? Explain.

(c) Suppose we delete strategies 1 and 10. That is, we rule out the possibility of any candidate choosing either 1 or 10, although there are still voters at those positions. Is strategy 2 dominated, strictly or weakly, by any other (pure) strategy s_i in the reduced game? Explain.

Solution

(i) Apart from strategy 2, strategies 3,4,5,6,7 strictly dominate strategy 1.

For the other three strategies 8,9,10, strategy 1 yields a higher percentage of votes if the opponent chooses strategy 7:

$$u_1(1,7) = 35\%; u_1(8,7) = 30\%; u_1(9,7) = 25\%; u_1(10,7) = 20\%$$

(ii) (a) weakly

$$u_1(1,2,3) = u_1(2,2,3) = 10\%$$

(b) weakly

$$u_1(1,2,4) = u_1(3,2,4) = 10\%$$

$$u_1(1,3,4) = u_1(3,3,4) = 15\%$$

(c) Not dominated. Strategy 2 is better than 3 against 2,4; better than 4 against 3,5; better than 5 against 4,6; better than 6 against 5,7; better than 7 against 6,8; better than 8 against 7,9; better than 9 against 9,9

Question 5: Bertrand Model with Homogeneous Products Consider 2 firms, labeled by $i = 1, 2$, selling homogeneous products in a market with unit demand. Suppose that firms' marginal costs are normalized to 0, and the firms set prices p_1 and p_2 simultaneously. Consumers purchase from the firm with a lower price p_i, provided that p_i is lower than their valuation. For simple exposition, we assume consumers are identical and have infinite valuation. The following tie-breaking rule is assumed: if two firms set the same price, each firm gets half of the market.

Therefore, firm i's payoff is as follows:

$$\pi_i = \begin{cases} p_i, & \text{if } p_i < p_j; \\ p_i/2, & \text{if } p_i = p_j; \\ 0, & \text{if } p_i > p_j. \end{cases}$$

(a) Show that there exists a **unique** pure strategy Nash equilibrium: both firms set $p = 0$.

(b) (Bonus question) Does there exist other (mixed strategy) Nash equilibria? (This is a difficult question. To start with, consider the symmetric case that each firm's pricing follow the distribution $p \sim G(p)$, where $G(p)$ is the cumulative distribution function.)

Solution Please refer to the file titled "Bertrand Competition".