Game Theory
Assignment 2 Solution

注：此答案步骤较简略，仅供参考。

Question 1: 3-stage Sequential Bargaining Consider the usual scenario that two players (Player 1 and Player 2) are bargaining over one bitcoin. They alternate in making offers. Player 1 discounts payoffs received in later periods by $\delta_1 < 1$ per period and player 2 discounts by $\delta_2 < 1$. The bargain will last for 3 periods at most, with the moves described as follows:

1. Player 1 makes an offer $(s_1, 1 - s_1)$.
   - If player 2 accepts the offer, then $(s_1, 1 - s_1)$ is the final division.
   - If player 2 rejects the offer, the game proceeds to Stage 2.
2. Player 2 makes an offer $(s_2, 1 - s_2)$.
   - If player 1 accepts the offer, then $(s_2, 1 - s_2)$ is the final division.
   - If player 1 rejects the offer, the game proceeds to Stage 3.
3. Player 1 makes an offer $(s_3, 1 - s_3)$.
   - If player 2 accepts the offer, then $(s_3, 1 - s_3)$ is the final division.
   - If player 2 rejects the offer, then the payoffs are $(0, 0)$.

What is the Backward Induction outcome of this game?

Solution

1. In stage 3, player 1 proposes $(1, 0)$; player 2 accepts.
2. In stage 2, player 2 proposes $(\delta_1, 1 - \delta_1)$; player 1 accepts.
3. In stage 1, player 1 proposes $(1 - \delta_2(1 - \delta_1), \delta_2(1 - \delta_1))$; player 2 accepts.

Question 2: Dating (Polak PS9 Q3) Alice and Bob are a couple. Alice can choose to stay at home reading a book or to go out with Bob. If Alice stays at home, her payoff is 3/4 while Bob's payoff is 0. If Alice chooses to go out, the couple would play the battle of the sexes game:

		Bob	
		Opera	Movie
Alice	Opera	$(k, 1)$	$(0, 0)$
	Movie	$(0, 0)$	$(1, 2)$

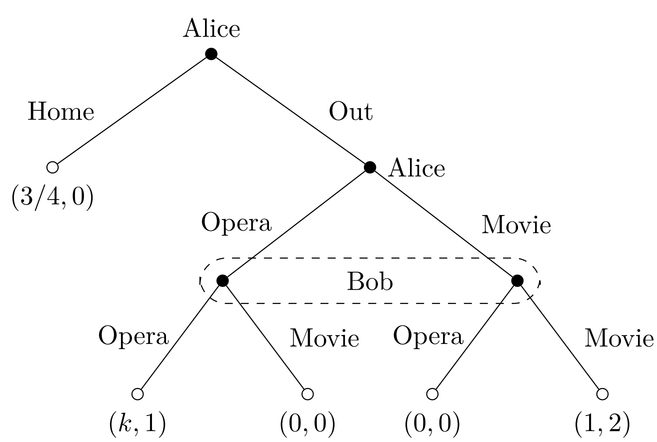
图 1: The Battle of the Sexes

where $k > 0$.

1. Draw the game tree (extensive form) and the matrix (normal form) of the game.
2. For what values of k, will *all* subgame perfect equilibria involve Alice going out? (Hint: Do not forget the mix-strategy Nash equilibrium in the subgame.)

Solution

1. Game tree:



Game matrix:

		Bob: Opera	Bob: Movie
Alice	[Home, Opera]	$(3/4, 0)$	$(3/4, 0)$
	[Home, Movie]	$(3/4, 0)$	$(3/4, 0)$
	[Out, Opera]	$(k, 1)$	$(0, 0)$
	[Out, Movie]	$(0, 0)$	$(1, 2)$

2. The battle of the sexes game is a subgame. This subgame has

   (a) two pure strategy NEs: $(Opera, Opera)$, $(Movie, Movie)$, and

   (b) one mixed strategy NE: $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{k+1}, \frac{k}{k+1}))$.

   Alice's payoffs from these three NEs are k, 1 and $\frac{k}{k+1}$ respectively. Since Alice obtains 3/4 when staying at home, to ensure that Alice would go out in any SPE, we require $k \geq 3/4$ and $\frac{k}{k+1} \geq 3/4$. So, $k \geq 3$.

Question 3: Gibbons 2.6 Three oligopolists operate in a market with inverse demand given by

$$P(Q) = a - Q,$$

where $Q = q_1 + q_2 + q_3$ and q_i is the quantity produced by firm i. Assume that the total cost to a firm with quantity q_i is $C(q_i) = cq_i$, where $c < a$. The firms choose their quantities as follows:

(i) firm 1 chooses $q_1 > 0$;

(ii) firms 2 and 3 observe q_1 and then simultaneously choose q_2 and q_3, respectively.

What is the subgame perfect outcome?

Solution In the second stage, given firm 1's quantity q_1, firms 2 and 3 engage in static Cournot competition. Their best response functions are

$$q_2(q_1) = q_3(q_1) = \frac{a - c - q_1}{3}. \tag{1}$$

In the first stage, given $q_2(q_1)$ and $q_3(q_1)$, firm 1 solves

$$\max_{q_1}(a - c - q_1 - q_2(q_1) - q_3(q_1))q_1.$$

First-order condition implies:
$$q_1^* = \frac{a-c}{2}.$$

Plugging q_1^* into (1):
$$q_2^* = q_3^* = \frac{a-c}{6}.$$

Question 4: Wars of Attrition (Polak PS9 Q1) Consider the two-period version of Wars of Attrition. Two players choose to "Fight (F)" or "Quit (Q)" in each period. The game ends as soon as at least one of the players chooses Q. Assume that there is no discounting. The payoffs to the players are as follows:

- If one of the players quits first, the player who does not quit win a prize v and the player who quits gets 0;
- If both players quit at once, both get 0;
- At each period in which both players choose F, each player pay a cost c. Assume $c > v$.

Solve for *all* subgame perfect Nash equilibrium of the game.

Solution The second-period game has

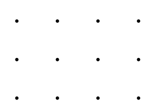
1. two pure strategy NEs: (F, Q), (Q, F), and
2. one mixed strategy NE: $((p_Q^* = \frac{c}{v+c}, p_F^* = \frac{v}{v+c}), (p_Q^*, p_F^*))$.

Consider the first-period game.

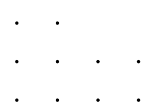
1. If the second-period outcome is (F, Q), we obtain 3 SPEs: $([Q, F], [F, Q])$, $([F, F], [Q, Q])$, $\left(\left[(\frac{c}{v+c}, \frac{v}{v+c}), F\right], \left[(\frac{c-v}{c}, \frac{v}{c}), Q\right]\right)$.
2. If the second-period outcome is (Q, F), we obtain 3 SPEs: $([Q, Q], [F, F])$, $([F, Q], [Q, F])$, $\left(\left[(\frac{c-v}{c}, \frac{v}{c}), Q\right], \left[(\frac{c}{v+c}, \frac{v}{v+c}), F\right]\right)$.
3. If the second-period outcome is $((p_Q^*, p_F^*), (p_Q^*, p_F^*))$, we obtain 3 SPEs: $([Q, (p_Q^*, p_F^*)], [F, (p_Q^*, p_F^*)])$、$([F, (p_Q^*, p_F^*)], [Q, (p_Q^*, p_F^*)])$, $([(p_Q^*, p_F^*), (p_Q^*, p_F^*)], [(p_Q^*, p_F^*), (p_Q^*, p_F^*)])$.

Bonus Question (Polak PS7 Q4) Initially, there is an $N \times M$ rectangle of stones. There are two players who take turns. The player whose turn it is to move must select a stone, removing that stone and the stones lie above or/and to the right of it. (每一轮，行动玩家选择一个未被移除的石头，则该石头以及该石头右上方 (包括右侧和上方) 所有未被移除的石头都被移除。) The loser is the person who takes the last stone.

A 3×4 example:



(i) If player 1 chooses the $(1, 3)$ stone, then the stones left are



(ii) If player 2 chooses the $(2, 2)$ stone, then the stones left are

(iii) If player 1 chooses the $(3, 3)$ stone, then the stones left are

(iv) If player 2 chooses the $(3, 1)$ stone, then all stones are removed. Player 2 loses (i.e., player 1 wins) the game.

By Zermelo's Theorem and that the game will never result in a tie, either player 1 can force a win or player 2 can force a loss on player 1. For all N and M, can you show whether player 1 can force a win or player 2 can force a loss on player 1?

Hint:

(i) $N = M = 1$ case is special.

(ii) For the other cases, you do not need to provide a full winning strategy.

Solution

(i) 当 $N = M = 1$ 时，显然玩家一必败。

(ii) 除去 $N = M = 1$ 情形，玩家一有必胜策略。

反证法。假设玩家二有必胜策略。也就是，所有移除一次后的残局都存在先手必胜策略。记移除一次后的残局所构成的集合为 S. 我们下面证明，在此假设下，玩家一第一步移除右上角的点, 即 $(1, M)$ 点，可以使玩家一必胜。

在玩家一移除 $(1, M)$ 点后，第二步无论玩家二选取哪个点，所构成的残局一定在集合 S 中。由于集合 S 中的所有情况都存在先手必胜策略，因此玩家一必胜。

玩家一存在必胜策略与原假设玩家二有必胜策略矛盾。

由于该游戏不存在平局，根据 Zermelo's Theorem, 玩家一存在必胜策略。证毕。