Game Theory
Assignment 3 Solution

注：此答案步骤较简略，仅供参考。

Question 1: Finitely Repeated Game (Polak PS10 Q2)

1. Find all pure-strategy Nash equilibria of the following game.

		Player 2			
		a	b	c	d
Player 1	A	$(3,1)$	$(0,0)$	$(0,0)$	$(5,0)$
	B	$(0,0)$	$(1,3)$	$(0,0)$	$(0,0)$
	C	$(0,0)$	$(0,0)$	$(2,2)$	$(0,0)$
	D	$(0,0)$	$(0,5)$	$(0,0)$	$(4,4)$

2. Suppose that the game is played twice. Assume no discounting. Construct a subgame perfect Nash equilibrium in which (D,d) is played in the first stage.

Solution

1. Pure strategy NE: (A,a), (B,b) and (C,c).

2. Consider the following strategy:

   (a) For player 1:

   - In the first stage, play D, and then
   - In the second stage,
     - Play B if (A, d) is played in the first stage;
     - Play A if (D, b) is played in the first stage;
     - Play C otherwise.

   (b) For player 2:

   - In the first stage, play d, and then

- In the second stage,
  - Play b if (A, d) is played in the first stage;
  - Play a if (D, b) is played in the first stage;
  - Play c otherwise.

Let us check that the above strategy profile constitute an SPE.

1. In the second stage, possible outcomes (B, b), (A, a) and (C, c) are all NE.
2. In the first stage, adding to the original payoffs the payoffs from the second stage, we obtain the following payoff matrix:

Player 2

Player 1	a	b	c	d
A	$(\underline{5}, \underline{3})$	$(2, 2)$	$(2, 2)$	$(\underline{6}, \underline{3})$
B	$(2, 2)$	$(\underline{3}, \underline{5})$	$(2, 2)$	$(2, 2)$
C	$(2, 2)$	$(2, 2)$	$(\underline{4}, \underline{4})$	$(2, 2)$
D	$(2, 2)$	$(\underline{3}, \underline{6})$	$(2, 2)$	$(\underline{6}, \underline{6})$

   (D, d) is indeed an NE in the first stage.

Question 2: Gibbons 2.15 Suppose there are n firms in a Cournot oligopoly. Inverse demand is given by

$$P(Q) = a - Q,$$

where $Q = q_1 + ... + q_n$ and q_i is the quantity produced by firm i. Assume that the total cost to a firm with quantity q_i is $C(q_i) = cq_i$, where $c < a$. Consider the infinitely repeated game based on this stage game.

1. What is the lowest value of δ such that the firms can use trigger strategies to sustain the monopoly output level in a subgame perfect Nash equilibrium?
2. How does the answer vary with n, and why?

3. If δ is too small for the firms to use trigger strategies to sustain the monopoly output, what is the most-profitable symmetric subgame perfect Nash equilibrium that can be sustained using trigger strategies?

Solution

1. It is straightforward to compute the monopoly price, and each firm's quantity and profit:

$$p^m = \frac{a+c}{2},\ q^m = \frac{a-c}{2n},\ \pi^m = \frac{(a-c)^2}{4n}.$$

Similarly, the Cournot equilibrium price, and each firm's quantity and profit are

$$p^c = \frac{a+nc}{n+1},\ q^c = \frac{a-c}{n+1},\ \pi^c = \frac{(a-c)^2}{(n+1)^2}.$$

If one of the firms deviates, it will choose quantity q^d by solving

$$\max_{q^d}(a-c-(n-1)q^m-q^d)q^d,$$

which implies

$$q^d = \frac{(n+1)(a-c)}{4n},\ \pi^d = \left[\frac{(n+1)(a-c)}{4n}\right]^2.$$

Therefore, the monopoly price and output can be sustained in the SPE if and only if

$$\frac{\pi^m}{1-\delta} \geq \pi^d + \delta\frac{\pi^c}{1-\delta},$$

which implies

$$\delta \geq \delta^* = \frac{(n+1)^2}{(n+1)^2+4n}.$$

2. The cutoff δ^* is strictly increasing in n. The intuition is as follows. The benefit from deviating is the first-period gain:

$$\pi^d - \pi^m;$$

whereas the loss from deviating is the perpetual reverting to Cournot outcome in the future:

$$\delta\frac{\pi^m - \pi^c}{1-\delta}.$$

When n is large, both π^m and π^c converges to 0. However, π^d converges to $\frac{(a-c)^2}{16}$. Therefore, collusion is harder to sustain when n is large.

3. Let $\hat{q}^e$ be each firm's production quantity in the most-profitable symmetric SPE. Then, on the equilibrium path, each firm's per-period profit is

$$\hat{\pi}^e = (a - c - n\hat{q}^e)\hat{q}^e.$$

If one of the firm deviates, its quantity and profit are

$$\hat{q}^d = \frac{a - c - (n-1)\hat{q}^e}{2}, \ \hat{\pi}^d = \left[\frac{a - c - (n-1)\hat{q}^e}{2}\right]^2.$$

Therefore, the output level $\hat{q}^e$ can be sustained in the SPE if and only if

$$\frac{\hat{\pi}^e}{1-\delta} \geq \hat{\pi}^d + \delta\frac{\pi^c}{1-\delta},$$

which implies

$$\hat{q}^e = \frac{[(n+1)^2 - \delta(n^2+2n-3)](a-c)}{(n+1)[(n+1)^2 - \delta(n-1)^2]}$$

Question 3: Gibbons 3.2 Consider the Cournot duopoly model where the two firms choose their quantities simultaneously. Let q_1 and q_2 denote the quantities (of a homogeneous product) produced by firms 1 and 2, respectively. Let $P(Q) = a - Q$ be the market-clearing price when the aggregate quantity on the market is $Q = q_1 + q_2$. The demand a is uncertain:

- it is high, i.e., $a = a_H$, with probability θ;
- it is low, i.e., $a = a_L (< a_H)$, with probability $1 - \theta$.

Assume that the total cost to a firm with quantity q_i is $C(q_i) = cq_i$.

The information is asymmetric:

- Firm 1 knows whether $a = a_H$ or $a = a_L$;
- Firm 2 only knows the prior distribution, i.e., $a = a_H$ w.p. θ and $a = a_L$ w.p. $1 - \theta$.

All of this is common knowledge. Assume that the parameters a_H, a_L, θ and c are such that all equilibrium quantities are positive.

1. What are the strategy spaces for the two firms?

2. What is the Bayesian Nash equilibrium of this game?

Solution

1. $S_1 = \{(q_{1H}, q_{1L}) | q_{1H} \geq 0, q_{1L} \geq 0\}$; $S_2 = \{q_2 | q_2 \geq 0\}$.

2. In the BNE, let firm 1's strategy be (q_{1H}^*, q_{1L}^*), and firm 2's strategy be q_2^*. For any $i \in \{H, L\}$, firm 1 solves

$$\max_{q_{1i}} (a_i - q_{1i} - q_2^* - c) q_{1i}.$$

First-order conditions imply

$$q_{1i}^* = \frac{a_i - c - q_2^*}{2}. \tag{1}$$

Firm 2 solves

$$\max_{q_2} \ \theta(a_H - q_{1H}^* - q_2 - c) q_2 + (1 - \theta)(a_L - q_{1L}^* - q_2 - c) q_2,$$

The first-order condition imply

$$q_2^* = \frac{\theta(a_H - q_{1H}^* - c) + (1 - \theta)(a_L - q_{1L}^* - c)}{2}. \tag{2}$$

From (1) and (2),

$$
\begin{aligned}
q_{1H}^* &= \frac{a_H - c}{3} + \frac{1-\theta}{6}(a_H - a_L); \\
q_{1L}^* &= \frac{a_L - c}{3} - \frac{\theta}{6}(a_H - a_L); \\
q_2^* &= \frac{\theta a_H + (1-\theta)a_L - c}{3}.
\end{aligned}
$$

Question 4: Gibbons 3.3 Consider the following asymmetric-information model of Bertrand duopoly with differentiated products. Demand for firm i is

$$q_i(p_i, p_j) = a - p_i - b_i \cdot p_j.$$

Costs are zero for both firms. The sensitivity of firm i's demand to firm j's price is either high or low. That is, b_i is either b_H or b_L, where $b_H > b_L > 0$. For each firm,

- $b_i = b_H$ with probability θ, and
- $b_i = b_L$ with probability $1-\theta$,

independent of the realization of b_j. Each him knows its own b_i, but not its competitor's. All of this is common knowledge.

1. What are the action spaces, type spaces, beliefs, and utility functions in this game?
2. What are the strategy spaces?
3. What conditions define a symmetric pure-strategy Bayesian Nash equilibrium of this game? Solve for such an equilibrium.

Solution

1. Action spaces: $A_1 = A_2 = [0, +\infty)$.

   Type spaces: $T_1 = T_2 = \{b_H, b_L\}$.

   Beliefs: For any $i, j \in \{1, 2\}$ and $i \neq j$, $\Pr(b_j = b_H | b_i) = \theta$, $\Pr(b_j = b_L | b_i) = 1 - \theta$.

Payoff functions: For any $i, j \in \{1, 2\}$, and $i \neq j$,

$$\pi_i(b_H) = \theta p_{iH}(a - p_{iH} - b_H p_{jH}) + (1-\theta)p_{iH}(a - p_{iH} - b_H p_{jL}),$$

$$\pi_i(b_L) = \theta p_{iL}(a - p_{iL} - b_L p_{jH}) + (1-\theta)p_{iL}(a - p_{iL} - b_L p_{jL}).$$

2. For any $i \in \{1, 2\}$, $S_i = \{(p_{iH}, p_{iL}) | p_{iH} \geq 0, p_{iL} \geq 0\}$.

3. For any $i, j \in \{1, 2\}$, and $i \neq j$, we have the following first-order conditions:

$$\frac{\partial \pi_i(b_H)}{p_{iH}} = 0, \ \frac{\partial \pi_i(b_L)}{p_{iL}} = 0,$$

which implies

$$p^*_{iH} = \frac{a[2 - (1-\theta)(b_H - b_L)]}{2[2 + \theta b_H + (1-\theta)b_L]}, \ p^*_{iL} = \frac{a[2 + \theta(b_H - b_L)]}{2[2 + \theta b_H + (1-\theta)b_L]}.$$

The symmetric pure-strategy BNE is $((p^*_{1H}, p^*_{1L}), (p^*_{2H}, p^*_{2L}))$.

Question 5: Gibbons 3.4 Find all the pure-strategy Bayesian Nash equilibria in the following static Bayesian game:

(i) Nature determines whether the payoffs are as in Game 1 or as in Game 2, each game being equally likely.

(ii) Player 1 learns whether nature has drawn Game 1 or Game 2, but player 2 does not.

(iii) Player 1 chooses either T or B; player 2 simultaneously chooses either L or R.

(iv) Payoffs are given by the game drawn by nature.

Game 1

Player 1 \ Player 2	L	R
T	(1,1)	(0,0)
B	(0,0)	(0,0)

Game 2

Player 1 \ Player 2	L	R
T	(0,0)	(0,0)
B	(0,0)	(2,2)

Solution Strategy spaces: $S_1 = \{[T, T], [T, B], [B, T], [B, B]\}$, $S_2 = \{L, R\}$.

1. Suppose player 2 chooses L, then player 1's best response is $[T, T]$ or $[T, B]$. Next, we check whether player 2's strategy L is a best response to $[T, T]$ and $[T, B]$.
   (a) Consider player 1's strategy $[T, T]$. Player 2 obtains 0.5 when choosing L and 0 when choosing R. Therefore, player 2's strategy L is indeed a best response.
   (b) Consider player 1's strategy $[T, B]$. Player 2 obtains 0.5 when choosing L and 1 when choosing R. Therefore, player 2's best response to $[T, B]$ is R.
2. Suppose player 2 chooses R, then player 1's best response is $[T, B]$ or $[B, B]$. Next, we check whether player 2's strategy R is a best response to $[T, B]$ and $[B, B]$.
   (a) Consider player 1's strategy $[T, B]$. Player 2 obtains 0.5 when choosing L and 1 when choosing R. Therefore, player 2's strategy R is indeed a best response.
   (b) Consider player 1's strategy $[B, B]$. Player 2 obtains 0 when choosing L and 1 when choosing R. Therefore, player 2's strategy R is indeed a best response.

Therefore, we obtain three pure strategy BNEs: $([T, T], L)$, $([T, B], R)$ and $([B, B], R)$