Advanced Microeconomics

Assignment 2 Solution

2.D.2 A consumer consumes one consumption good x and hours of leisure h. The price of the consumption good is p, and the consumer can work at a wage rate of $s = 1$. What is the consumer's Walrasian budget set?

Solution. The price of leisure is the wage rate, since otherwise the consumer could utlize the time to work and enjoy the wage. Total wealth is the number of hours endowed (24) times the wage rate. So the consumer's Walrasian budget set is as follows:
$\left\{(x, h) \in \mathbb{R}_+^2 : h \leq 24,\ px + h \leq 24\right\}$.

2.E.1 Suppose $L = 3$, and consider the demand function $x\,(p, w)$ defined by

$$
\begin{aligned}
x_1\,(p, w) &= \frac{p_2}{p_1 + p_2 + p_3}\frac{w}{p_1} \\
x_2\,(p, w) &= \frac{p_3}{p_1 + p_2 + p_3}\frac{w}{p_2} \\
x_3\,(p, w) &= \frac{\beta p_1}{p_1 + p_2 + p_3}\frac{w}{p_3}
\end{aligned}
$$

Does this demand function satisfy homogeneity of degree zero and Walras' law when $\beta = 1$? What about when $\beta \in (0, 1)$?

Solution. When $\beta = 1$,

$$
\begin{aligned}
x_1\,(\alpha p, \alpha w) &= \frac{\alpha p_2}{\alpha p_1 + \alpha p_2 + \alpha p_3}\frac{\alpha w}{\alpha p_1} = \frac{p_2}{p_1 + p_2 + p_3}\frac{w}{p_1} = x_1\,(p, w) \\
x_2\,(\alpha p, \alpha w) &= \frac{\alpha p_3}{\alpha p_1 + \alpha p_2 + \alpha p_3}\frac{\alpha w}{\alpha p_2} = \frac{p_3}{p_1 + p_2 + p_3}\frac{w}{p_2} = x_2\,(p, w) \\
x_3\,(\alpha p, \alpha w) &= \frac{\alpha p_1}{\alpha p_1 + \alpha p_2 + \alpha p_3}\frac{\alpha w}{\alpha p_3} = \frac{p_1}{p_1 + p_2 + p_3}\frac{w}{p_3} = x_3\,(p, w)
\end{aligned}
$$

So homogeneity of degree zero is satisfied.

$$\begin{aligned}
p \cdot x\left(p,w\right) &= p_1 \cdot \frac{p_2}{p_1+p_2+p_3}\frac{w}{p_1} + p_2 \cdot \frac{p_3}{p_1+p_2+p_3}\frac{w}{p_2} + p_3 \cdot \frac{p_1}{p_1+p_2+p_3}\frac{w}{p_3} \\
&= \frac{p_2}{p_1+p_2+p_3}w + \frac{p_3}{p_1+p_2+p_3}w + \frac{p_1}{p_1+p_2+p_3}w \\
&= w
\end{aligned}$$

So Walras' law is satisfied.

When $\beta \in (0,1)$,

$$\begin{aligned}
x_1\left(\alpha p,\alpha w\right) &= \frac{\alpha p_2}{\alpha p_1+\alpha p_2+\alpha p_3}\frac{\alpha w}{\alpha p_1} = \frac{p_2}{p_1+p_2+p_3}\frac{w}{p_1} = x_1\left(p,w\right) \\
x_2\left(\alpha p,\alpha w\right) &= \frac{\alpha p_3}{\alpha p_1+\alpha p_2+\alpha p_3}\frac{\alpha w}{\alpha p_2} = \frac{p_3}{p_1+p_2+p_3}\frac{w}{p_2} = x_2\left(p,w\right) \\
x_3\left(\alpha p,\alpha w\right) &= \frac{\alpha\beta p_1}{\alpha p_1+\alpha p_2+\alpha p_3}\frac{\alpha w}{\alpha p_3} = \frac{\beta p_1}{p_1+p_2+p_3}\frac{w}{p_3} = x_3\left(p,w\right)
\end{aligned}$$

So homogeneity of degree zero is satisfied.

$$\begin{aligned}
p \cdot x\left(p,w\right) &= p_1 \cdot \frac{p_2}{p_1+p_2+p_3}\frac{w}{p_1} + p_2 \cdot \frac{p_3}{p_1+p_2+p_3}\frac{w}{p_2} + p_3 \cdot \frac{\beta p_1}{p_1+p_2+p_3}\frac{w}{p_3} \\
&= \frac{p_2}{p_1+p_2+p_3}w + \frac{p_3}{p_1+p_2+p_3}w + \frac{\beta p_1}{p_1+p_2+p_3}w \\
&= \frac{\beta p_1+p_2+p_3}{p_1+p_2+p_3}w \neq w
\end{aligned}$$

So Walras' law is not satisfied.

2.E.3 Use Proposition 2.E.1 to 2.E.3 to show that $p \cdot D_p x\left(p,w\right)p = -w$.

Solution. Recall Proposition 2.E.2:

$$p \cdot D_p x(p,w) + x(p,w)^T = 0^T. \tag{1}$$

Multiply both sides of (1) on the right by p gives

$$\begin{aligned}
&p \cdot D_p x(p,w)p + x(p,w)^T p = 0 \\
\Rightarrow &p \cdot D_p x(p,w)p + w = 0 \text{ by Walras' law} \\
\Rightarrow &p \cdot D_p x(p,w)p = -w
\end{aligned}$$

2.E.5 Suppose that $x(p, w)$ is a demand function which is homogeneous of degree one with respect to w and satisfies Walras' law and homogeneity of degree zero. Suppose also that all the cross-price effects are zero, that is $\partial x_l(p, w)/\partial p_k = 0$ whenever $k \neq l$. Show that this implies that for every l, $x_l(p, w) = \alpha_l w/p_l$, where $\alpha_l > 0$ is a constant independent of (p, w).

Solution. H.D.1 with respect to w :

$$x(p, \alpha w) = \alpha x(p, w)$$

Set $\alpha = \frac{1}{w}$, we have

$$\begin{aligned} x(p, 1) &= \frac{1}{w} x(p, w) \Rightarrow x(p, w) = w \cdot x(p, 1) \\ \Rightarrow & x_l(p, w) = w x_l(p, 1). \end{aligned}$$

Since $\partial x_l(p, w)/\partial p_k = 0$, $x_l(p, w)$ is a function of p_l alone, i.e.,

$$x_l(p_l, w) = w x_l(p_l).$$

H.D.∅:

$$\begin{aligned} x(\alpha p, \alpha w) &= x(p, w) \Rightarrow x_l(\alpha p_l, \alpha w) = x_l(p_l, w) \\ \Rightarrow & \alpha w x_l(\alpha p_l) = w x_l(p_l) \Rightarrow x_l(\alpha p_l) = \frac{1}{\alpha} x_l(p_l) \end{aligned}$$

So $x_l(p_l)$ is homogeneous of degree -1 with respect to p_l. Hence there exists $\alpha_l > 0$ such that

$$x_l(p_l) = \frac{\alpha_l}{p_l}.$$ [1]

So,

$$x_l(p, w) = \frac{\alpha_l w}{p_l}.$$

By Walras' law,

$$\begin{aligned} p \cdot x(p, w) = w &\Rightarrow \sum_{l=1}^{L} p_l \cdot x_l(p, w) = w \\ \Rightarrow \sum_{l=1}^{L} p_l \frac{\alpha_l w}{p_l} = w &\Rightarrow \sum_{l=1}^{L} \alpha_l w = w \Rightarrow \sum_{l=1}^{L} \alpha_l = 1. \end{aligned}$$

[1] More explicitly, since $x_l(\alpha p_l) = \frac{1}{\alpha} x_l(p_l)$ for any $\alpha > 0$, we could set $\alpha = \frac{1}{p_l}$. Then $x_l(1) = p_l x_l(p_l) \implies x_l(p_l) = \frac{x_l(1)}{p_l}$. Setting $\alpha_l = x_l(1)$ gives the result.

2.E.7 A consumer in a two-good economy has a demand function $x(p, w)$ that satisfies Walras' law. His demand function for the first good is $x_1(p, w) = \alpha w / p_1$. Derive his demand function for the second good. Is his demand function homogeneous of degree zero?

Solution. Walras' law:

$$\begin{aligned} p_1 x_1 + p_2 x_2 = w &\Rightarrow p_1 \frac{\alpha w}{p_1} + p_2 x_2 = w \\ \Rightarrow p_2 x_2 = (1-\alpha) w &\Rightarrow x_2 = \frac{(1-\alpha) w}{p_2}. \end{aligned}$$

To check H.D.$\emptyset$, we need to show $x_1(\gamma p, \gamma w) = x_1(p, w)$ and $x_2(\gamma p, \gamma w) = x_2(p, w)$.

$$\begin{aligned} x_1(\gamma p, \gamma w) &= \frac{\alpha \gamma w}{\gamma p_1} = \frac{\alpha w}{p_1} = x_1(p, w) \\ x_2(\gamma p, \gamma w) &= \frac{(1-\alpha) \gamma w}{\gamma p_2} = \frac{(1-\alpha) w}{p_2} = x_2(p, w) \end{aligned}$$

So homogeneity of degree zero is satisfied.

2.E.8 Show that the elasticity of demand for good l with respect to price p_k, $\varepsilon_{lk}(p, w)$, can be written as $\varepsilon_{lk}(p, w) = d \ln(x_l(p, w)) / d \ln(p_k)$, where $\ln(\cdot)$ is the natural logarithm function. Derive a similar expression for $\varepsilon_{lw}(p, w)$. Conclude that if we estimate the parameters $(\alpha_0, \alpha_1, \alpha_2, \gamma)$ of the equation $\ln(x_l(p, w)) = \alpha_0 + \alpha_1 \ln p_1 + \alpha_2 \ln p_2 + \gamma \ln w$, these parameter estimates provide us with estimates of the elasticities $\varepsilon_{l1}(p, w)$, $\varepsilon_{l2}(p, w)$, and $\varepsilon_{lw}(p, w)$.

Solution.

$$\begin{aligned} \frac{d \ln(x_l(p, w))}{d \ln(p_k)} &= \frac{d(x_l(p, w)) / (x_l(p, w))}{d(p_k) / (p_k)} = \varepsilon_{lk}(p, w) \\ \frac{d \ln(x_l(p, w))}{d \ln w} &= \frac{d(x_l(p, w)) / (x_l(p, w))}{d(w) / (w)} = \varepsilon_{lw}(p, w) \end{aligned}$$

$$\begin{aligned} \alpha_1 &= \frac{d \ln(x_l(p, w))}{d \ln p_1} = \varepsilon_{l1}(p, w) \\ \alpha_2 &= \frac{d \ln(x_l(p, w))}{d \ln p_2} = \varepsilon_{l2}(p, w) \\ \gamma &= \frac{d \ln(x_l(p, w))}{d \ln w} = \varepsilon_{lw}(p, w) \end{aligned}$$

2.F.11 Show that for $L = 2$, $S(p, w)$ is always symmetric. [Hint: Use Proposition 2.F.3.]

Solution.

$$S(p,w) = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \text{ and } p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

From Proposition 2.F.3, $p \cdot S(p,w) = 0$ and $S(p,w)\, p = 0$

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \cdot \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = 0$$

$$\Rightarrow \quad p_1 s_{11} + p_2 s_{21} = 0 \tag{2}$$

$$p_1 s_{12} + p_2 s_{22} = 0$$

$$\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 0$$

$$\Rightarrow \quad p_1 s_{11} + p_2 s_{12} = 0 \tag{3}$$

$$p_1 s_{21} + p_2 s_{22} = 0$$

From (2) and (3), $p_1 s_{11} + p_2 s_{21} = p_1 s_{11} + p_2 s_{12} \Rightarrow s_{21} = s_{12}$

2.F.17 In an L-commodity world, a consumer's Walrasian demand function is

$$x_k(p,w) = \frac{w}{\sum_{l=1}^{L} p_l} \text{ for } k = 1, ..., L.$$

(a) In this demand function homogeneous of degree zero in (p, w)?

(b) Does it satisfy Walras' law?

(c) Does it satisfy the weak axiom?

(d) Compute the Slutsky substitution matrix for this demand function. Is it negative semidefinite? Negative definite? Symmetric?

Solution.

(a)
$$x_k(\alpha p, \alpha w) = \frac{\alpha w}{\sum_{l=1}^{L} \alpha p_l} = \frac{w}{\sum_{l=1}^{L} p_l} = x_k(p,w)$$

So homogeneity of degree zero is satisfied.

(b) $$p \cdot x(p,w) = \sum_{k=1}^{L} p_k x_k(p,w) = \sum_{k=1}^{L} p_k \frac{w}{\sum_{l=1}^{L} p_l} = w \frac{\sum_{k=1}^{L} p_k}{\sum_{l=1}^{L} p_l} = w$$

So Walras' law is satisfied.

(c) Check $p \cdot x(p', w') \leq w$ and $x(p,w) \neq x(p',w') \Rightarrow p' \cdot x(p,w) > w'$.

$$p \cdot x(p',w') = \sum_{k=1}^{L} p_k x_k(p',w') = \sum_{k=1}^{L} p_k \frac{w'}{\sum_{l=1}^{L} p'_l} = w' \frac{\sum_{k=1}^{L} p_k}{\sum_{l=1}^{L} p'_l} \leq w$$

then

$$p' \cdot x(p,w) = \sum_{k=1}^{L} p'_k x_k(p,w) = \sum_{k=1}^{L} p'_k \frac{w}{\sum_{l=1}^{L} p_l} = w \frac{\sum_{k=1}^{L} p'_k}{\sum_{l=1}^{L} p_l} \geq w'$$

Suppose

$$w \frac{\sum_{k=1}^{L} p'_k}{\sum_{l=1}^{L} p_l} = w'$$

then

$$\frac{w}{\sum_{l=1}^{L} p_l} = \frac{w'}{\sum_{k=1}^{L} p'_k} \Rightarrow x_k(p,w) = x_k(p',w') \text{ for } k = 1, ..., L$$

Contradicting with

$$x(p,w) \neq x(p',w')$$

So

$$p' \cdot x(p,w) > w'$$

holds. And WARP is satisfied.

(d) $$S_{lk} = \frac{\partial x_l}{\partial p_k} + \frac{\partial x_l}{\partial w} x_k = \frac{-w}{\left(\sum_{l=1}^{L} p_l\right)^2} + \frac{1}{\sum_{l=1}^{L} p_l} \frac{w}{\sum_{l=1}^{L} p_l} = 0$$

So

$$S(p,w) = 0.$$

It is symmetric and negative semi-definite, but NOT negative definite.