Advanced Microeconomics

Assignment 3 Solution

3.B.2 The preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}_+^L$ is said to be *weakly monotone* if and only if $x \geq y$ implies that $x \succsim y$. Show that if $\succsim$ is transitive, locally nonsatiated, and weakly monotone, then it is monotone.

Solutions. We want to show that if $x \gg y$, then $x \succ y$. Define $\varepsilon = \min\{x_1 - y_1, x_2 - y_2, ..., x_L - y_L\}$, then $\forall z \in X$, if $\|z - y\| < \varepsilon$, then $x \gg z$. By local nonsatiation, $\exists z^* \in X$, such that $\|z - y\| < \varepsilon$ and $z^* \succ y$. On the other hand, $x \succsim z^*$ due to weak monotonicity. Hence, by transitivity, we have $x \succ y$.

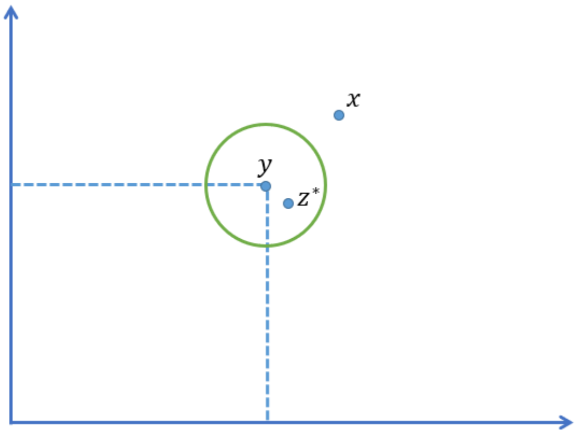

Figure 1: 3.B.2

3.C.6 Suppose that in a two-commodity world, the consumer's utility function takes the form $u(x) = [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho}$. This utility function is known as the *constant elasticity of substitution* (or *CES*) utility function.

(a) Show that when $\rho = 1$, indifference curves become linear.

(b) Show that as $\rho \to 0$, this utility function comes to represent the same preference as the (generalized) Cobb-Douglas utility function $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$.

(c) Show that as $\rho \to -\infty$, indifference curves become "right angles"; that is, this utility function has in the limit the indifference map of the Leontief utility function $u(x_1, x_2) = \min\{x_1, x_2\}$.

Solution.

(a) When $\rho = 1, u(x) = \alpha_1 x_1 + \alpha_2 x_2$. Take the total differentiation of the utility function, we have $du(x) = \alpha_1 dx_1 + \alpha_2 dx_2$. On an indifference curve, $du(x) = 0$, and hence $\alpha_1 dx_1 + \alpha_2 dx_2 = 0$. Rearranging terms gives

$$\frac{dx_2}{dx_1} = -\frac{\alpha_1}{\alpha_2},$$

which implies that the indifference curve is a straight line.

(b) Consider $\hat{u}(x) := \ln(u(x)) = (1/\rho)\ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)$, which represents the same preference as $u(x)$ does. By L'Hôpital's rule (To use L'Hôpital's rule, we require $\alpha_1 + \alpha_2 = 1$.), we have

$$\begin{aligned}\lim_{\rho\to 0}\hat{u}(x) &= \lim_{\rho\to 0}\frac{\ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)}{\rho} \\ &= \lim_{\rho\to 0}\frac{\alpha_1 x_1^\rho \ln(x_1) + \alpha_2 x_2^\rho \ln(x_2)}{\alpha_1 x_1^\rho + \alpha_2 x_2^\rho} \\ &= \frac{\alpha_1 \ln(x_1) + \alpha_2 \ln(x_2)}{\alpha_1 + \alpha_2} \equiv \tilde{u}(x).\end{aligned}$$

Since $\exp[(\alpha_1 + \alpha_2)\tilde{u}(x)] = x_1^{\alpha_1} x_2^{\alpha_2}$, we obtain the Cobb-Douglas utility function.

Alternative way to show that the utility functions represent the same preference: we check whether the two utility functions have the same *marginal rate of substitution*. If they have the same marginal rate of substitution, the consumer who is described by either utility function is willing to make the same trade-offs. That is, the two utility functions represent the same preference. (This alternative solution does not require $\alpha_1 + \alpha_2 = 1$.)

For CES utility function, we first transform $\hat{u}(x) := \ln(u(x)) = (1/\rho)\ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)$, then

$$\frac{\partial \hat{u}(x)}{\partial x_1} = \frac{1}{\rho}\frac{\alpha_1 \rho x_1^{\rho-1}}{\alpha_1 x_1^\rho + \alpha_2 x_2^\rho} \text{ and } \frac{\partial \hat{u}(x)}{\partial x_2} = \frac{1}{\rho}\frac{\alpha_2 \rho x_2^{\rho-1}}{\alpha_1 x_1^\rho + \alpha_2 x_2^\rho}$$
$$\Longrightarrow \frac{\frac{\partial \hat{u}(x)}{\partial x_1}}{\frac{\partial \hat{u}(x)}{\partial x_2}} = \frac{\alpha_1}{\alpha_2}\left[\frac{x_1}{x_2}\right]^{\rho-1} \Longrightarrow \lim_{\rho\to 0}\frac{\frac{\partial \hat{u}(x)}{\partial x_1}}{\frac{\partial \hat{u}(x)}{\partial x_2}} = \frac{\alpha_1 x_2}{\alpha_2 x_1}$$

For Cobb-Douglas utility function $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$,

$$\frac{\partial u(x)}{\partial x_1} = \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} \text{ and } \frac{\partial u(x)}{\partial x_2} = \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1}$$
$$\implies \frac{\frac{\partial u(x)}{\partial x_1}}{\frac{\partial u(x)}{\partial x_2}} = \frac{\alpha_1 x_2}{\alpha_2 x_1}$$

The two utility functions share the same marginal rate of substitution, so the two utilities represent the same preference.

(c) Without loss of generality, we assume $x_1 \leq x_2$, then it suffices to show that $\lim_{\rho \to -\infty} [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho} = x_1$.

Consider $\rho < 0$. Since $x_1 \leq x_2$, we have

$$[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho} \geq [\alpha_1 x_1^\rho + \alpha_2 x_1^\rho]^{1/\rho},$$

i.e., $u(x) \geq (\alpha_1 + \alpha_2)^{1/\rho} x_1$.

Hence,

$$\lim_{\rho \to -\infty} u(x) \geq \lim_{\rho \to -\infty} (\alpha_1 + \alpha_2)^{1/\rho} x_1 = x_1. \tag{1}$$

On the other hand, since $x_1, x_2 \geq 0$, we have

$$[\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho} \leq [\alpha_1 x_1^\rho]^{1/\rho},$$

i.e., $u(x) \leq \alpha_1^{1/\rho} x_1$.

Hence,

$$\lim_{\rho \to -\infty} u(x) \leq \alpha_1^{1/\rho} x_1 = x_1. \tag{2}$$

Therefore, by (1) and (2), we have $\lim_{\rho \to -\infty} u(x) = x_1$.

3.D.5 Consider again the CES utility function of Exercise 3.C.6, and assume that $\alpha_1 = \alpha_2 = 1$.

(a) Compute the Walrasian demand and indirect utility functions for this utility function.

(b) Verify that these functions satisfy all the properties of Propositions 3.D.2 and 3.D.3.

(c) Derive the Walrasian demand correspondence and indirect utility function for the case of linear utility and the case of Leontief utility (see Exercise 3.C.6). Show that the CES Walrasian demand and indirect utility functions approach these as ρ approaches 1 and $-\infty$, respectively.

(d) The *elasticity of substitution between goods* 1 and 2 is defined as

$$\xi_{12}(p, w) = -\frac{\partial[x_1(p, w)/x_2(p, w)]}{\partial[p_1/p_2]} \frac{p_1/p_2}{x_1(p, w)/x_2(p, w)}.$$

Show that for the CES utility function, $\xi_{12}(p, w) = 1/(1 - \rho)$, thus justifying the name. What is $\xi_{12}(p, w)$ for the linear, Leontief, and Cobb-Douglas utility functions?

Solution.

(a) Formulate the utility maximization problem as follows:

$$\max_{x_1, x_2 \geq 0} u(x) = (x_1^\rho + x_2^\rho)^{1/\rho},$$
$$\text{s.t. } p_1 x_1 + p_2 x_2 \leq w.$$

The constraint should hold in equality at the optimum, since any wealth left could have been spent to increase utility.

Set up the Lagrangian

$$\mathcal{L}(x_1, x_2, \lambda) = (x_1^\rho + x_2^\rho)^{1/\rho} - \lambda(p_1 x_1 + p_2 x_2 - w).$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{1}{\rho}(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1}(\rho x_1^{\rho-1}) - \lambda p_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= \frac{1}{\rho}(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}-1}(\rho x_2^{\rho-1}) - \lambda p_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= p_1 x_1 + p_2 x_2 - w = 0
\end{aligned}$$

The first two FOCs imply that

$$\frac{x_1}{x_2} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}.$$

Together with the third FOC, one can solve for the Walrasian demand function, which is given by

$$x(p,w) = \left(\frac{wp_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}, \frac{wp_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}\right).$$

Substitute $x(p,w)$ back into the utility function, we obtain the indirect utility function

$$\begin{aligned} v(p,w) &= \left[\left(\frac{wp_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}\right)^\rho + \left(\frac{wp_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}\right)^\rho\right]^{1/\rho} \\ &= \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} w. \end{aligned}$$

(b) **Proposition 3.D.2**

(i) **Homogeneity of degree zero** of the demand function. For any p, w and $\alpha > 0$, we have

$$x_1(\alpha p, \alpha w) = \frac{(\alpha p_1)^{\frac{1}{\rho-1}}}{(\alpha p_1)^{\frac{\rho}{\rho-1}} + (\alpha p_2)^{\frac{\rho}{\rho-1}}}\alpha w = \frac{p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w = x_1(p,w),$$

$$x_2(\alpha p, \alpha w) = \frac{(\alpha p_2)^{\frac{1}{\rho-1}}}{(\alpha p_1)^{\frac{\rho}{\rho-1}} + (\alpha p_2)^{\frac{\rho}{\rho-1}}}\alpha w = \frac{p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w = x_2(p,w).$$

(ii) **Walras' law**. Direct calculation gives

$$\begin{aligned} p_1x_1 + p_2x_2 &= p_1\frac{p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w + p_2\frac{p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w \\ &= \frac{p_1^{\frac{\rho}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w + \frac{p_2^{\frac{\rho}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w \\ &= w, \end{aligned}$$

(iii) The **uniqueness** is trivial: $x(p,w)$ is unique given the explicit expression.

Proposition 3.D.3

(i) **Homogeneity of degree zero** in price of the indirect utility function. For any $\alpha > 0$, we have

$$\begin{aligned} v(\alpha p, w) &= \left((\alpha p_1)^{\frac{\rho}{\rho-1}} + (\alpha p_2)^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} w \\ &= \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} w \\ &= v(p, w). \end{aligned}$$

(ii) **Monotonicity** of the indirect utility function. Since for any $p \gg 0, w > 0$,

$$\begin{aligned} \frac{\partial v(p,w)}{\partial w} &= \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} > 0, \\ \frac{\partial v(p,w)}{\partial p_l} &= \frac{1-\rho}{\rho}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-2\rho}{\rho}} w \left(\frac{\rho}{\rho-1} p_l^{\frac{1}{\rho-1}}\right) \\ &= -\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-2\rho}{\rho}} w p_l^{\frac{1}{\rho-1}} < 0 \text{ for } l = 1, 2. \end{aligned}$$

Hence, this indirect function is strictly increasing in w and strictly decreasing in p_l for all l.

(iii) **Quasiconvexity**. To prove quasiconvexity, we claim that, by homogeneity of degree zero, it suffices to prove that for any $\bar{v} \in \mathbb{R}$ and $w > 0$, the set $\{p \in \mathbb{R}^2_{++} : v(p,w) \le \bar{v}\}$ is convex. (Proved later)

For $\rho \to 0$, the utility function is Cobb-Douglas, and the indirect utility function is given by $v(p,w) = \frac{w^2}{4p_1p_2}$ which is convex in p [1], and hence quasiconvex. For $\rho < 0$, since $p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}$ is concave in p [2], the set $\{p : v(p,w) \le \bar{v}\} = \{p : p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}} \ge \bar{v}^{\frac{\rho}{1-\rho}}\}$ is convex.

[1] The Hessian matrix of the function $v(p,w) = \frac{w^2}{4p_1p_2}$ is $w^2 \begin{bmatrix} \frac{1}{2p_1^3 p_2} & \frac{1}{4p_1^2p_2^2} \\ \frac{1}{4p_1^2p_2^2} & \frac{1}{2p_1p_2^3} \end{bmatrix}$, which is positive definite when $p \gg 0$.

[2] The Hessian matrix of the function $g(p) = p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}$ is $\begin{bmatrix} \frac{\rho}{(\rho-1)^2} p_1^{\frac{2-\rho}{\rho-1}} & 0 \\ 0 & \frac{\rho}{(\rho-1)^2} p_2^{\frac{2-\rho}{\rho-1}} \end{bmatrix}$, which is negative definite when $\rho < 0$ and $p \gg 0$.

For $\rho \in (0,1)$, since $p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}$ is convex in p [3], the set $\{p : v(p,w) \leq \bar{v}\} = \{p : p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}} \leq \bar{v}^{\frac{\rho}{1-\rho}}\}$ is convex.

Now we prove the claim:

Claim. To prove quasiconvexity, by homogeneity of degree zero, it suffices to prove that for any $\bar{v} \in \mathbb{R}$ and $w > 0$, the set $\{p \in \mathbb{R}^2_{++} : v(p,w) \leq \bar{v}\}$ is convex.

Note that quasiconvexity states that

$$v(p,w) \leq \bar{v}, v(p',w') \leq \bar{v} \implies v(\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w') \leq \bar{v}.$$

And homogeneity of degree zero implies $v(p,w) = v\left(\frac{p}{w}, 1\right)$. Hence, quasiconvexity is equivalent to

$$v\left(\frac{p}{w}, 1\right) \leq \bar{v}, v\left(\frac{p'}{w'}, 1\right) \leq \bar{v} \implies v\left(\frac{\alpha p + (1-\alpha)p'}{\alpha w + (1-\alpha)w'}, 1\right) \leq \bar{v}. \tag{3}$$

Let $p_1 = \frac{p}{w}$, $p_2 = \frac{p'}{w'}$, then

$$\frac{\alpha p + (1-\alpha)p'}{\alpha w + (1-\alpha)w'} = \beta p_1 + (1-\beta)p_2,$$

where

$$\beta = \frac{\alpha w}{\alpha w + (1-\alpha)w'} \in (0,1).$$

Now if the set $\{p \in \mathbb{R}^L_{++} : v(p,w) \leq \bar{v}\}$ is convex, we have

$$v(p_1, 1) \leq \bar{v}, v(p_2, 1) \leq \bar{v} \implies v(\beta p_1 + (1-\beta)p_2, 1) \leq \bar{v}.$$

Therefore, we have established (3) by the convexity of the set $\{p \in \mathbb{R}^L_{++} : v(p,w) \leq \bar{v}\}$.

(iv) **Continuity** follows from the functional form of $v(p,w)$.

[3]The Hessian matrix of the function $g(p) = p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}$ is $\begin{bmatrix} \frac{\rho}{(\rho-1)^2} p_1^{\frac{2-\rho}{\rho-1}} & 0 \\ 0 & \frac{\rho}{(\rho-1)^2} p_2^{\frac{2-\rho}{\rho-1}} \end{bmatrix}$, which is positive definite when $\rho \in (0,1)$ and $p \gg 0$.

(c) For linear utility function, $u(x) = x_1 + x_2$, one can solve for the Walrasian demand function by substituting $x_2 = \frac{w - p_1 x_1}{p_2}$ into the objective function, which gives

$$x(p,w) = \begin{cases} (w/p_1, 0), & \text{if } p_1 < p_2; \\ (0, w/p_2), & \text{if } p_2 < p_1; \\ (w/p_1)(\lambda, 1-\lambda), \lambda \in [0,1], & \text{if } p_1 = p_2. \end{cases}$$

And the indirect utility function is given by

$$v(p,w) = \max(w/p_1, w/p_2).$$

Similarly, for Leontief utility function $u(x) = \min\{x_1, x_2\}$, the Walrasian demand function is given by

$$x(p,w) = (\frac{w}{p_1 + p_2}, \frac{w}{p_1 + p_2}).$$

And the indirect utility function is given by

$$v(p,w) = \frac{w}{p_1 + p_2}.$$

Consider $\rho < 1$ and $\rho \to 1$.

If $p_1 < p_2$, $(p_2/p_1)^{\frac{\rho}{\rho-1}} \to 0$, $(p_1/p_2)^{\frac{\rho}{\rho-1}} \to \infty$ as $\rho \to 1^-$. Thus,

$$\begin{aligned} \lim_{\rho \to 1^-} x_1(p,w) &= \lim_{\rho \to 1^-} \frac{p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w \\ &= \lim_{\rho \to 1^-} \frac{p_1^{-1}}{1 + (p_2/p_1)^{\frac{\rho}{\rho-1}}} w \\ &= w/p_1, \end{aligned}$$

and

$$\begin{aligned} \lim_{\rho \to 1^-} x_2(p,w) &= \lim_{\rho \to 1^-} \frac{p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} w \\ &= \lim_{\rho \to 1^-} \frac{p_2^{-1}}{(p_1/p_2)^{\frac{\rho}{\rho-1}} + 1} w \\ &= 0. \end{aligned}$$

Similarly, if $p_1 > p_2$,

$$\lim_{\rho \to 1^-} x_1(p, w) = 0,$$

$$\lim_{\rho \to 1^-} x_2(p, w) = w/p_2.$$

Finally, if $p_1 = p_2$, then

$$\lim_{\rho \to 1^-} x_l(p, w) = \frac{w}{2p_1}, \text{ for } l = 1, 2.$$

Therefore, the CES Walrasian demands converge to the Walrasian demand of the linear preference as $\rho \to 1$.

As for the indirect utility function, if $p_1 < p_2$, then

$$\begin{aligned}
\lim_{\rho \to 1^-} v(p, w) &= \lim_{\rho \to 1^-} \left(p_1^{\frac{\rho}{\rho - 1}} + p_2^{\frac{\rho}{\rho - 1}} \right)^{\frac{1-\rho}{\rho}} w \\
&= \lim_{\rho \to 1^-} \left(1 + (p_2/p_1)^{\frac{\rho}{\rho - 1}} \right)^{\frac{1-\rho}{\rho}} \frac{w}{p_1} \\
&= \frac{w}{p_1}.
\end{aligned}$$

If $p_1 > p_2$, then

$$\begin{aligned}
\lim_{\rho \to 1^-} v(p, w) &= \lim_{\rho \to 1^-} \left(p_1^{\frac{\rho}{\rho - 1}} + p_2^{\frac{\rho}{\rho - 1}} \right)^{\frac{1-\rho}{\rho}} w \\
&= \lim_{\rho \to 1^-} \left((p_1/p_2)^{\frac{\rho}{\rho - 1}} + 1 \right)^{\frac{1-\rho}{\rho}} \frac{w}{p_2} \\
&= \frac{w}{p_2}.
\end{aligned}$$

If $p_1 = p_2$, then

$$\begin{aligned}
\lim_{\rho \to 1^-} v(p, w) &= \lim_{\rho \to 1^-} \left(p_1^{\frac{\rho}{\rho - 1}} + p_2^{\frac{\rho}{\rho - 1}} \right)^{\frac{1-\rho}{\rho}} w \\
&= \lim_{\rho \to 1^-} 2^{\frac{1-\rho}{\rho}} \frac{w}{p_1} \\
&= \frac{w}{p_1},
\end{aligned}$$

which belongs to the set of the Walrasian demands of the linear preference when $p_1 = p_2$.

Therefore, $\lim_{\rho \to 1^-} v(p, w) = \max(w/p_1, w/p_2)$ which agrees with the indirect utility function of the linear preference.

Now consider the case when $\rho \to -\infty$. Since $\frac{\rho}{\rho-1} \to 1$, we have

$$\begin{aligned}\lim_{\rho\to-\infty} x_1(p,w) &= \lim_{\rho\to-\infty} \frac{p_1^{-1}}{1+(p_2/p_1)^{\frac{\rho}{\rho-1}}} w \\ &= \frac{w}{p_1+p_2},\end{aligned}$$

and

$$\begin{aligned}\lim_{\rho\to-\infty} x_2(p,w) &= \lim_{\rho\to-\infty} \frac{p_2^{-1}}{(p_1/p_2)^{\frac{\rho}{\rho-1}}+1} w \\ &= \frac{w}{p_1+p_2}.\end{aligned}$$

Also,

$$\begin{aligned}\lim_{\rho\to-\infty} v(p,w) &= \lim_{\rho\to-\infty} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{1-\rho}{\rho}} w \\ &= \frac{w}{p_1+p_2}.\end{aligned}$$

Therefore, the CES Walrasian demand function and the indirect utility function converge to those of the Leontief preference as $\rho \to -\infty$.

(d) Recall that the FOC of the utility maximization problem with CES utility is

$$\frac{x_1}{x_2} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}.$$

Hence, the elasticity of substitution between goods 1 and 2 is

$$\begin{aligned}\xi_{12}(p,w) &= -\frac{\partial[x_1(p,w)/x_2(p,w)]}{\partial[p_1/p_2]} \frac{p_1/p_2}{x_1(p,w)/x_2(p,w)} \\ &= -\frac{1}{\rho-1}\left(\frac{p_1}{p_2}\right)^{\frac{2-\rho}{\rho-1}} \frac{p_1}{p_2}\left(\frac{p_1}{p_2}\right)^{\frac{-1}{\rho-1}} \\ &= \frac{1}{1-\rho},\end{aligned}$$

which is a constant.

Similarly, we have $\xi_{12}(p,w) = \lim_{\rho\to1^-} \frac{1}{1-\rho} = \infty$ for linear utility, $\xi_{12}(p,w) = \lim_{\rho\to-\infty} \frac{1}{1-\rho} = 0$ for Leontief utility, and $\xi_{12}(p,w) = \lim_{\rho\to0} \frac{1}{1-\rho} = 1$ for Cobb-Douglas utility.

3.E.6 Consider the constant elasticity of substitution utility function studied in Exercises 3.C.6 and 3.D.5 with $\alpha_1 = \alpha_2 = 1$. Derive its Hicksian demand function and expenditure function. Verify the properties of Propositions 3.E.2 and 3.E.3.

Solution. Consider the following expenditure minimization problem:

$$\begin{aligned}\min_{x_1,x_2\geq 0} &\ p_1x_1 + p_2x_2,\\ \text{s.t. } &u(x) = (x_1^\rho + x_2^\rho)^{1/\rho} \geq u.\end{aligned}$$

The constraint should be binding at optimum since otherwise a sufficiently small reduction in consumption can reduce expenditure without violating the constraint. We then set up the Lagrangian as follows:

$$\mathcal{L}(x_1, x_2, \lambda) = -p_1x_1 - p_2x_2 - \lambda[-(x_1^\rho + x_2^\rho)^{1/\rho} + u].$$

The first-order conditions are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1} &= -p_1 + \lambda\rho^{-1}(x_1^\rho + x_2^\rho)^{1/\rho-1}\rho x_1^{\rho-1} = 0,\\ \frac{\partial \mathcal{L}}{\partial x_2} &= -p_2 + \lambda\rho^{-1}(x_1^\rho + x_2^\rho)^{1/\rho-1}\rho x_2^{\rho-1} = 0,\\ \frac{\partial \mathcal{L}}{\partial \lambda} &= u - (x_1^\rho + x_2^\rho)^{1/\rho} = 0.\end{aligned}$$

From the first two FOCs, we obtain $\frac{x_1}{x_2} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}$. Together with the third FOC, we derive the Hicksian demand function as follows:

$$\begin{aligned}h_1(p,u) &= up_1^{\frac{1}{\rho-1}}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}},\\ h_2(p,u) &= up_2^{\frac{1}{\rho-1}}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}}.\end{aligned}$$

The expenditure function is thus given by

$$e(p,u) = p\cdot h(p,u) = u\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}.$$

We check the properties of the expenditure function as follows.

Proposition 3.E.2

(i) For any $\alpha > 0$, we have

$$\begin{aligned} e(\alpha p, u) &= u\left((\alpha p_1)^{\frac{\rho}{\rho-1}} + (\alpha p_2)^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} \\ &= \alpha u \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} \\ &= \alpha e(p, u). \end{aligned}$$

Hence, the expenditure function is **homogeneous of degree one in p.**

(ii) Since for any $u > 0$ and $p \gg 0$

$$\frac{\partial e(p, u)}{\partial u} = \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} > 0,$$

and

$$\frac{\partial e(p, u)}{\partial p_l} = u\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} p_l^{\frac{1}{\rho-1}} > 0, \text{ for } l = 1, 2,$$

this expenditure function is **strictly increasing in u and p_l for all l.**

(iii) Since the Hessian matrix

$$D_p^2 e(p, u) = \begin{bmatrix} \frac{u}{\rho-1} p_1^{\frac{2-\rho}{\rho-1}} p_2^{\frac{\rho}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1+\rho}{\rho}} & -\frac{u}{\rho-1}(p_1 p_2)^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1+\rho}{\rho}} \\ -\frac{u}{\rho-1}(p_1 p_2)^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1+\rho}{\rho}} & \frac{u}{\rho-1} p_2^{\frac{2-\rho}{\rho-1}} p_1^{\frac{\rho}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1+\rho}{\rho}} \end{bmatrix}$$

is negative definite, this expenditure function is **strictly concave in p.**

(iv) The **continuity** of $e(p, u)$ is trivial.

We check the properties of the Hicksian demand function as follows.

Proposition 3.E.3

(i) Since for any $\alpha > 0$ and $l = 1, 2$,

$$\begin{aligned} h_l(\alpha p, u) &= u(\alpha p_l)^{\frac{1}{\rho-1}} \left((\alpha p_1)^{\frac{\rho}{\rho-1}} + (\alpha p_2)^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} \\ &= u p_1^{\frac{1}{\rho-1}} \left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} \\ &= h_l(p, u), \end{aligned}$$

the Hicksian demand function is **homogeneous of degree zero in p.**

(ii) Since

$$u(h_1(p,u), h_2(p,u)) = \left\{\left[up_1^{\frac{1}{\rho-1}}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}}\right]^{\rho} + \left[up_2^{\frac{1}{\rho-1}}\left(p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}}\right]^{\rho}\right\}^{1/\rho}$$
$$= u,$$

there is **no excess utility**.

(iii) The **uniqueness** of the value of Hicksian demand is trivial.

3.E.9 Use the relations in (3.E.1) to show that the properties of the indirect utility function identified in Proposition 3.D.3 imply Proposition 3.E.2. Likewise, use the relations in (3.E.1) to prove that Proposition 3.E.2 implies Proposition 3.D.3.

Solution. We first show that by relations in (3.E.1), the properties of the indirect utility function in Proposition 3.D.3 imply the properties of the expenditure function in Proposition 3.E.2.

Let $p \gg 0, p' \gg 0, u \in \mathbb{R}, u' \in \mathbb{R}$, and $\alpha \geq 0$.

(i) **Homogeneity:** Let $\alpha > 0$. Define $w = e(p,u)$, then $u = v(p,w)$ by the second relation in (3.E.1). Hence,

$$e(\alpha p, u) = e(\alpha p, v(p,w)) = e(\alpha p, v(\alpha p, \alpha w)) = \alpha w = \alpha e(p,u),$$

where the second equality follows from the homogeneity of $v(p,w)$ and the third from the first relation in (3.E.1). Therefore, the expenditure function is homogeneous of degree one in p.

(ii) **Monotonicity:** Let $u' > u$. Define $w = e(p,u)$ and $w' = e(p,u')$, then $u = v(p,w)$ and $u' = v(p,w')$. Since $v(p,w)$ is strictly increasing in w, we must have $w' > w$, that is, $e(p,u') > e(p,u)$.

Next let $p' \geq p$. Define $w = e(p,u)$ and $w' = e(p',u)$, then by the second relation in (3.E.1), $u = v(p,w) = v(p',w')$. Since $v(p,w)$ is strictly increasing in w and nonincreasing in p_l for any l, we must have $w' \geq w$, that is, $e(p',u) \geq e(p,u)$. Therefore, the expenditure function $e(p,u)$ is strictly increasing in u and nondecreasing in p_l for any l.

(iii) **Concavity:** Let $\alpha \in [0,1]$. Define $w = e(p,u)$ and $w' = e(p',u)$, then $u = v(p,w) = v(p',w')$. Define $p'' = \alpha p + (1-\alpha)p'$ and $w'' = \alpha w + (1-\alpha)w'$. Then, by the quasiconvexity of $v(p,w)$, $v(p'',w'') \leq u$. Hence, since $v(p,w)$ is strictly increasing in w and $v(p'', e(p'',u)) = u$, we must have $e(p'',u) \geq w''$, that is,

$$e(\alpha p + (1-\alpha)p', u) \geq \alpha e(p,u) + (1-\alpha)e(p',u).$$

Therefore, the expenditure function $e(p,u)$ is concave in p.

(iv) **Continuity:** Suppose the sequence $\{(p^n,u^n)\}_{n=1}^{\infty}$ converges to (p,u), we show that $\lim_{n\to\infty} e(p^n,u^n) = e(p,u)$. Suppose to the contrary that $\lim_{n\to\infty} e(p^n,u^n) = w \neq e(p,u)$ for some $w \in \mathbb{R}$.

- On the one hand, by the second relation in (3.E.1), $v(p^n, e(p^n,u^n)) = u^n$, which converges to u by assumption.
- On the other hand, since $v(p,w)$ is continuous in (p,w) and is strictly increasing in w, $\lim_{n\to\infty} e(p^n,u^n) = w \neq e(p,u)$ implies that $\lim\limits_{n\to\infty} v(p^n, e(p^n,u^n)) = v(p,w) \neq v(p, e(p,u)) = u$.

We reach a contradiction. Hence, we must have $\lim_{n\to\infty} e(p^n,u^n) = e(p,u)$, i.e., $e(p,u)$ is continuous in (p,u).

Now we show that by relations in (3.E.1), the properties of the expenditure function in Proposition 3.E.2 imply the properties of the indirect utility function in Proposition 3.D.3.

Let $p \gg 0, p' \gg 0, w \in \mathbb{R}, w' \in \mathbb{R}$, and $\alpha \geq 0$.

(i) **Homogeneity:** Let $\alpha > 0$. Define $u = v(p,w)$, then by the first relation in (3.E.1), $e(p,u) = w$. Hence,

$$v(\alpha p, \alpha w) = v(\alpha p, \alpha e(p,u)) = v(\alpha p, e(\alpha p, u)) = u = v(p,w),$$

where the second equality follows from the homogeneity of $e(p,u)$ and the third from the second relation in (3.E.1). Therefore, the indirect utility function $v(p,w)$ is homogeneous of degree zero.

(ii) **Monotonicity:** Let $w' > w$. Define $u = v(p, w)$ and $u' = v(p, w')$, then $e(p, u) = w$ and $e(p, u') = w'$. Since $e(p, u)$ is strictly increasing in u, we must have $u' > u$, that is, $v(p, w') > v(p, w)$. Therefore, the indirect utility function is strictly increasing in w.

Next let $p' \geq p$. Define $u = v(p, w)$ and $u' = v(p', w)$, then $e(p, u) = e(p', u') = w$. Since $e(p, u)$ is strictly increasing in u and nondecreasing in p_l for any l, we must have $u' \leq u$, that is $v(p', w) \leq v(p, w)$. Therefore, the indirect utility function is nonincreasing in p.

(iii) **Quasiconvexity:** Let $\alpha \in [0, 1]$. Define $u = v(p, w)$ and $u' = v(p', w')$, then $e(p, u) = w$ and $e(p', u') = w'$. Without loss of generality, assume that $u' \geq u$. Define $p'' = \alpha p + (1-\alpha)p'$ and $w'' = \alpha w + (1-\alpha)w'$, and we show that $v(p'', w'') \leq u'$. Since $u' = v(p'', e(p'', u'))$ and $v(p, w)$ is strictly increasing in w, it suffices to show that $e(p'', u') \geq w''$. This is proved as follows:

$$\begin{aligned} e(p'', u') &\geq \alpha e(p, u') + (1-\alpha) e(p', u') \\ &\geq \alpha e(p, u) + (1-\alpha) e(p', u') \\ &= \alpha w + (1-\alpha) w' = w'', \end{aligned}$$

where the first inequality follows from the concavity of $e(p, u)$ in p, and the second from the monotonicity of $e(p, u)$ in u. Therefore, the indirect utility function $v(p, w)$ is quasiconvex.

(iv) **Continuity:** Suppose the sequence $\{(p^n, w^n)\}_{n=1}^{\infty}$ converges to (p, w), we show that $\lim_{n\to\infty} v(p^n, w^n) = v(p, w)$. Suppose to the contrary that $\lim_{n\to\infty} v(p^n, w^n) = u \neq v(p, w)$ for some $u \in \mathbb{R}$.

- On the one hand, $e(p^n, v(p^n, w^n)) = w^n$, which converges to w by assumption.
- On the other hand, as $e(p, u)$ is continuous in (p, u) and is strictly increasing in u, $e(p^n, v(p^n, w^n))$ converges to $e(p, u) \neq e(p, v(p, w)) = w$.

We reach a contradiction. Hence, we must have $\lim_{n\to\infty} v(p^n, w^n) = v(p, w)$, i.e., $v(p, w)$ is continuous in (p, w).

3.G.1 Prove that Proposition 3.G.1 is implied by Roy's identity (Proposition 3.G.4).

Solution. Define $w = e(p,u)$, then $v(p,w) = u$. Differentiate both sides of $v(p, e(p,u)) = u$ with respect to p, and we have

$$\nabla_p v(p,w) + \nabla_w v(p,w) \nabla_p e(p,u) = 0.$$

By Roy's identity, $\nabla_p v(p,w) = -x(p,w)\nabla_w v(p,w)$, we have

$$\begin{aligned} &- x(p,w)\nabla_w v(p,w) + \nabla_w v(p,w)\nabla_p e(p,u) = 0, \\ \Longrightarrow\ &\nabla_w v(p,w)[\nabla_p e(p,u) - x(p, e(p,u))] = 0. \end{aligned}$$

Since $\nabla_w v(p,w) > 0$ and $x(p, e(p,u)) = h(p,u)$, we obtain $h(p,u) = \nabla_p e(p,u)$.

3.G.8 The indirect utility function $v(p,w)$ is logarithmically homogeneous if $v(p, \alpha w) = v(p,w) + \ln \alpha$ for $\alpha > 0$ [in other words, $v(p,w) = \ln(v^*(p,w))$, where $v^*(p,w)$ is homogeneous of degree one in w]. Show that if $v(\cdot,\cdot)$ is logarithmically homogeneous, then $x(p,1) = -\nabla_p v(p,1)$.

Solution. For any $w > 0$, by Roy's identity, we have

$$x(p,w)\nabla_w v(p,w) = -\nabla_p v(p,w).$$

Since $v(\cdot,\cdot)$ is logarithmically homogeneous, we rewrite the above equation as

$$x(p,w)\nabla_w (v(p,1) + \ln(w)) = -\nabla_p v(p,w).$$

Hence,

$$x(p,w)/w = -\nabla_p v(p,w).$$

Evaluate at $w = 1$, we obtain

$$x(p,1) = -\nabla_p v(p,1).$$

3.G.15 Consider the utility function

$$u = 2x_1^{1/2} + 4x_2^{1/2}.$$

(a) Find the demand functions for goods 1 and 2 as they depend on prices and wealth.

(b) Find the compensated demand function $h(\cdot)$.

(c) Find the expenditure function, and verify that $h(p,u) = \nabla_p e(p,u)$.

(d) Find the indirect utility function, and verify Roy's identity.

Solution.

(a) Consider the following utility maximization problem

$$\max_{x_1,x_2\geq 0} u(x) = 2x_1^{1/2} + 4x_2^{1/2},$$
$$\text{s.t. } p_1x_1 + p_2x_2 \leq w.$$

The budget constraint should hold in equality at the optimum since the marginal utility of each good is always positive for any positive consumption.

We set up the Lagrangian

$$\mathcal{L}(x,\lambda) = 2x_1^{1/2} + 4x_2^{1/2} - \lambda[p_1x_1 + p_2x_2 - w].$$

The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= x_1^{-1/2} - \lambda p_1 = 0, \\
\frac{\partial \mathcal{L}}{\partial x_2} &= 2x_2^{-1/2} - \lambda p_2 = 0, \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= p_1x_1 + p_2x_2 - w = 0.
\end{aligned}$$

The first two FOCs imply

$$\frac{x_2}{x_1} = \left(\frac{2p_1}{p_2}\right)^2.$$

Together with the third FOC, we can derive the Walrasian demand function

$$x(p,w) = \left(\frac{p_2 w}{p_1 p_2 + 4p_1^2}, \frac{4p_1 w}{4p_1 p_2 + p_2^2}\right).$$

(b) Consider the expenditure minimization problem

$$\begin{aligned} \min_{x_1, x_2 \geq 0} \ & p_1 x_1 + p_2 x_2, \\ \text{s.t. } & u(x) = 2x_1^{1/2} + 4x_2^{1/2} \geq u. \end{aligned}$$

The constraint must hold in equality at the optimum since otherwise any sufficiently small reduction in consumption can reduce expenditure without violating the utility requirement.

Hence, we set up the Lagrangian as follows:

$$\mathcal{L}(x,\lambda) = -p_1 x_1 - p_2 x_2 - \lambda[-2x_1^{1/2} - 4x_2^{1/2} + u].$$

The first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= -p_1 + \lambda x_1^{-1/2} = 0, \\ \frac{\partial \mathcal{L}}{\partial x_2} &= -p_2 + \lambda 2 x_2^{-1/2} = 0, \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= 2x_1^{1/2} + 4x_2^{1/2} - u = 0. \end{aligned}$$

The first two FOCs imply that

$$\frac{x_2}{x_1} = \left(\frac{2p_1}{p_2}\right)^2.$$

Together with the third FOC, we obtain the Hicksian demand function

$$h(p,u) = \left(\left(\frac{up_2}{8p_1 + 2p_2}\right)^2, \left(\frac{up_1}{4p_1 + p_2}\right)^2\right).$$

(c) The expenditure function is given by

$$e(p,u) = p_1 \left(\frac{up_2}{8p_1 + 2p_2}\right)^2 + p_2 \left(\frac{up_1}{4p_1 + p_2}\right)^2 = \frac{p_1 p_2 u^2}{4(4p_1 + p_2)}.$$

Direct calculation gives

$$\begin{aligned}\frac{\partial e(p,u)}{\partial p_1} &= \frac{4p_2u^2(4p_1+p_2)-16p_1p_2u^2}{16(4p_1+p_2)^2}\\ &= \left(\frac{up_2}{8p_1+2p_2}\right)^2 = h_1(p,u),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial e(p,u)}{\partial p_2} &= \frac{4p_1u^2(4p_1+p_2)-4p_1p_2u^2}{16(4p_1+p_2)^2}\\ &= \left(\frac{up_1}{4p_1+p_2}\right)^2 = h_2(p,u).\end{aligned}$$

Hence, $h(p,u) = \nabla_p e(p,u)$.

(d) The indirect utility function is given by

$$v(p,w) = 2\left(\frac{p_2w}{p_1p_2+4p_1^2}\right)^{1/2} + 4\left(\frac{4p_1w}{p_2^2+4p_1p_2}\right)^{1/2}.$$

Direct calculation gives

$$\frac{\partial v(p,w)}{\partial w} = \left(\frac{p_2}{p_1p_2+4p_1^2}\right)^{1/2} w^{-1/2} + \left(\frac{16p_1}{p_2^2+4p_1p_2}\right)^{1/2} w^{-1/2} = \frac{(p_2+4p)^{1/2}}{(p_1p_2)^{1/2}}w^{-1/2},$$

and

$$\begin{aligned}\frac{\partial v(p,w)}{\partial p_1} &= \left(\frac{p_2w}{p_1(p_2+4p_1)}\right)^{-1/2} \frac{-(p_2+8p_1)p_2w}{[p_1(p_2+4p_1)]^2}\\ &\quad + 2\left(\frac{4p_1w}{p_2(p_2+4p_1)}\right)^{-1/2} \frac{4wp_2(p_2+4p_1)-4p_2\cdot 4w}{[p_2(p_2+4p_1)]^2}\\ &= \frac{-(p_2+8p_1)(p_2w)^{1/2}}{[p_1(p_2+4p_1)]^{3/2}} + \frac{(p_1w)^{-1/2}4wp_2^2}{[p_2(p_2+4p_1)]^{3/2}}\\ &= \frac{-(p_2+8p_1)(p_2w)^{1/2} + (p_1w)^{-1/2}p_1^{3/2}4wp_2^{1/2}}{p_1^{3/2}(p_2+4p_1)^{3/2}}\\ &= \frac{-p_2^{3/2}w^{1/2} - 4p_1p_2^{1/2}w^{1/2}}{p_1^{3/2}(p_2+4p_1)^{3/2}}\\ &= \frac{-(p_2w)^{1/2}}{p_1^{3/2}(p_2+4p_1)^{1/2}}.\end{aligned}$$

Therefore,

$$-\frac{\partial v(p,w)/\partial p_1}{\partial v(p,w)/\partial w} = -\frac{-(p_2 w)^{1/2}}{p_1^{3/2}(p_2+4p_1)^{1/2}} \cdot \frac{(p_1 p_2)^{1/2}}{(p_2+4p)^{1/2}} w^{1/2}$$
$$= \frac{p_2 w}{p_1(p_2+4p_1)} = x_1(p,w).$$

Similarly, one can check that

$$-\frac{\partial v(p,w)/\partial p_2}{\partial v(p,w)/\partial w} = x_2(p,w).$$

Therefore, Roy's identity holds.

Additional Exercise

Claim 2. A function $f\colon \mathbb{R}^L :\to \mathbb{R}$ is continuous if and only if for all a, the set $\{x \in \mathbb{R}^L : f(x) \geq a\}$ and the set $\{x \in \mathbb{R}^L : f(x) \leq a\}$ are both closed.

Prove the "only if" part of the claim above.

Solution.

Function continuous. A function $f\colon \mathbb{R}^L \to \mathbb{R}$ is said to be continuous at a point $x \in \mathbb{R}^L$ if for any sequence of points $\{x^n\}_{n=1}^{\infty}$ converging to x (i.e., $x = \lim_{n\to\infty} x^n$), the sequence $f(x^n)$ converges to $f(x)$ (i.e., $f(x) = \lim_{n\to\infty} f(x^n)$).

Set $\{x \in \mathbb{R}^L : f(x) \geq a\}$ closed. The set $\{x \in \mathbb{R}^L : f(x) \geq a\}$ is closed if for any sequence of points $\{x^n\}_{n=1}^{\infty}$ converging to x with $\{x^n \in \mathbb{R}^L : f(x^n) \geq a\}$ for all n, we have $\{x \in \mathbb{R}^L : f(x) \geq a\}$.

Consider the sequence $\{x^n\}_{n=1}^{\infty}$ with $x = \lim_{n\to\infty} x^n$. Suppose $f(x^n) \geq a$ for all n, then $\lim_{n\to\infty} f(x^n) \geq a$. By continuity of f, $\lim_{n\to\infty} f(x^n) = f(x)$. Thus, $f(x) \geq a$.

Closedness of the set $\{x \in \mathbb{R}^L : f(x) \leq a\}$ can be similarly proved.