Chapter 1. Preference and Choice

1.A. Introduction

Two approaches to modeling individual choice behavior:

1. Preference-based Approach: preference as primative (rationality axioms) $\Longrightarrow$ consequences on choices

2. Choice-based Approach: choice behavior as primative (axioms on behavior)

Traditional: Preference-based Approach is preferred.
Some attractive features of Choice-based Approach: allows more room for general forms of behavior, assumptions on observable behavior, doesn't require introspection

1.B. Preference Relations

X: Set of Alternatives. For example, if Alice just graduated from Wuhan University majoring in economics, then her set of alternatives is: $X = \{$go to graduate school and study economics, go to a Big-4 firm, go to work for the government, ..., run a small business$\}$.
We use capital letters (like X and B) for a set of alternatives, small letters (like x and y) for a specific choice alternative.

Defining Preference Relations Denote by $\succsim$ *the preference relation* defined on the set X, allowing the comparison of any x and y in X.
$x \succsim y$: pronounced as "x is preferred to y" or "x is at least as good as y." The first usage is more common.
Strict preference $\succ$: $x \succ y \iff x \succsim y$ but not $y \succsim x$ (i.e., $y \not\succsim x$) ("x is strictly preferred to y.")
Indifference $\sim$: $x \sim y \iff x \succsim y$ and $y \succsim x$ ("x is indifferent to y.")

Rational Preference Not all preference relations make sense. For example, consider that Alice strictly prefers "Hot and Dry Noodles" to "Doupi" (dòu pí), strictly prefers

"Doupi" to "Xiaolongbao" (xiǎo lóng bāo), and strictly prefers "Xiaolongbao" to "Hot and Dry Noodles." Alice must have a hard time choosing her breakfast from $X = \{$Hot and Dry Noodles, Doupi, Xiaolongbao$\}$.

Definition 1.B.1 (Rational preference)**.** The preference relation $\succsim$ is **rational** if it possesses these two properties:

(i) Completeness: $\forall x, y \in X$, $x \succsim y$ or $y \succsim x$. (rules out $x \not\succsim y$ and $y \not\succsim x$)

(ii) Transitivity: $\forall x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Question 1. In the example above, which property does Alice's preference relation violate?

Answer: Transitivity. Proof by contradiction. Suppose *Transitivity* holds. Since strict preference implies weak preference, Alice prefers "Hot and Dry Noodles" to "Doupi" and "Doupi" to "Xiaolongbao," by transitivity, she must prefer "Hot and Dry Noodles" to "Xiaolongbao." This contradicts that she strictly prefers "Xiaolongbao" to "Hot and Dry Noodles."

Exercise (additional)

Question. Can you think of an example in which the preference relation is transitive but not complete?

Implications on $\succ$ and $\sim$ The following propositions follow from the definition of *rational preference.*

Proposition 1.B.1. *If $\succsim$ is rational, then:*

(i) $\succ$ is both irreflexive ($x \succ x$ never holds) and transitive.

(ii) $\sim$ is reflexive ($x \sim x$), transitive and symmetric (if $x \sim y$, then $y \sim x$).

(iii) if $x \succ y \succsim z$, then $x \succ z$. (slightly stronger than transitivity in (i))

Proof.

(i) *Irreflexive.* Proof by contradiction. Suppose $x \succ x$, then

$$x \succsim x \text{ and } x \not\succsim x \quad \text{(definition of } \succ\text{)},$$

which is never true.

Transitive. Proof by contradiction. Suppose $x \succ y, y \succ z$ and $z \succsim x$, then

$$y \succ z \implies y \succsim z \quad \text{(definition of } \succ\text{)},$$

and

$$y \succsim z \;\&\; z \succsim x \implies y \succsim x \quad \text{(transitivity of } \succsim\text{)}.$$

This contradicts that $x \succ y$.

(ii) *Reflexive.* By completeness of $\succsim$, $x \succsim x$. Then, $x \sim x$ by definition of $\sim$.

Transitive. Suppose $x \sim y, y \sim z$, then

$$\begin{aligned} & x \succsim y, y \succsim z \;\&\; y \succsim x, z \succsim y \quad \text{(definition of } \sim\text{)} \\ \implies & x \succsim z, z \succsim x \quad \text{(transitivity of } \succsim\text{)} \\ \implies & x \sim z \quad \text{(definition of } \sim\text{)} \end{aligned}$$

Symmetric. Suppose $x \sim y$, then $x \succsim y$ and $y \succsim x$ (definition of $\sim$). Using the definition of $\sim$ again, $y \sim x$.

(iii) Proof by contradiction. Suppose $x \succ y, y \succsim z$ and $z \succsim x$, then

$$y \succsim z \;\&\; z \succsim x \implies y \succsim x \quad \text{(transitivity of } \succsim\text{)}.$$

This contradicts that $x \succ y$. □

Utility Functions It seems unnecessarily abstract to use always the preference relation $\succsim$. Since human beings are better at comparing the order of numbers, we assign each choice with a number. In doing that, we are using some *utility function* to represent the preference relation.

Definition 1.B.2. A function $u : X \to \mathbb{R}$ is a utility function representing preference relation $\succsim$ if

$$x \succsim y \iff u(x) \geq u(y) \text{ for all } x, y \in X. \tag{1}$$

The utility function is nothing but assigning each choice x with a number $u(x)$. Obviously, the function u satisfying Condition (1) is not unique.

Example. $u(x) \geq u(y) \iff \alpha u(x) \geq \alpha u(y)$ for all $\alpha > 0$.

Exercise 1.B.3

Show that if $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function and $u : X \to \mathbb{R}$ is a utility function representing preference relation $\succsim$, then the function $v : X \to \mathbb{R}$ defined by $v(x) = f(u(x))$ is also a utility function representing preference relation $\succsim$.

Question 2. When can a preference relation be represented by a utility function?

Answer: Only if the preference relation is rational. See the next proposition.

Proposition 1.B.2. *If the preference relation $\succsim$ can be represented by a utility function (i.e. $\exists u(\cdot)$ s.t. $u(x) \geq u(y)$ iff $x \succsim y$), then $\succsim$ is rational (i.e. complete & transitive).*

Proof. Suppose there exists some $u(\cdot)$ such that $u(x) \geq u(y)$ iff $x \succsim y$.

Completeness: $u(x), u(y) \in \mathbb{R} \implies u(x) \geq u(y)$ or $u(y) \geq u(x) \iff x \succsim y$ or $y \succsim x$

Transitivity: $x \succsim y$ & $y \succsim z \iff u(x) \geq u(y)$ & $u(y) \geq u(z) \implies u(x) \geq u(z) \iff x \succsim z$. □

Question 3. If $\succsim$ is rational, does there exist a utility function u representing $\succsim$?

Answer: Not always. Rationality is just a necessary condition for the existence of a utility representation, but not sufficient. See the counterexample below.

Definition (Lexicographic Preference)**.** Let $X = \mathbb{R}^2$. The preference relation $\succsim$ is a *lexicographic preference* if for all $x, y \in X$, $x \succsim y$ whenever (i) $x_1 > y_1$ or (ii) $x_1 = y_1$ and $x_2 \geq y_2$.

Claim. The lexicographic preference on $\mathbb{R}^2$ do *not* have a utility representation.

Let's look at an example of the lexicographic preference before moving into the proof. As the lexicographic preference is defined on $\mathbb{R}^2$, it is used to describe a decision making situation with *two-dimensional comparisons.* For example, Alice is considering buying a new phone. The relevant attributes include brand name, price, CPU, and so on. For simplicity, suppose Alice only cares about the price and the brand (Apple or Huawei)[1]. Alice's first priority is the price. (Of course, Alice prefers low price to high price.) At the same price, Alice prefers an iPhone to a Huawei Phone. For Example,

$$(5000, \text{Huawei}) \succ (8000, \text{Apple}) \succ (8000, \text{Huawei}).$$

Alice's desicion making criteria satisfy the requirements of the lexicographic preference. Although her preference is rational, it cannot be modelled by a utility function.

Proof.

1. Lexicographic Preference is rational.

   Completeness. For $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$:

   a) If $x_1 > y_1$, then $x \succsim y$

   b) If $y_1 > x_1$, then $y \succsim x$

   c) If $x_1 = y_1$, then either $x_2 \geq y_2 \implies x \succsim y$ or $y_2 \geq x_2 \implies y \succsim x$

   Transitivity. Let $x, y, z \in \mathbb{R}^2$. Suppose $x \succsim y$ and $y \succsim z$. Then, one of the following cases must prevail:

   a) $x_1 > y_1$ and $y_1 > z_1$

   b) $x_1 > y_1$, $y_1 = z_1$ and $y_2 \geq z_2$

   c) $x_1 = y_1$, $x_2 \geq y_2$ and $y_1 > z_1$

   d) $x_1 = y_1$, $x_2 \geq y_2$, $y_1 = z_1$ and $y_2 \geq z_2$

   In each case, $x \succsim z$ since

   a) $x_1 > y_1 > z_1 \implies x_1 > z_1$

   b) $x_1 > y_1 = z_1 \implies x_1 > z_1$

[1] In this example, the preference is defined on $\mathbb{R}_+ \times \{Apple, Huawei\}$.

c) $x_1 = y_1 > z_1 \implies x_1 > z_1$

d) $x_1 = y_1 = z_1$ and $x_2 \geq y_2 \geq z_2 \implies x_1 = z_1$ and $x_2 \geq z_2$

2. There does not exist $u(\cdot)$ that represents Lexicographic Preference.

   We prove by contradiction. Suppose $\exists u(\cdot)$ that represents $\succsim$.

   For any $x_1 \in \mathbb{R}$, $u(x_1, 2) > u(x_1, 1)$ (definition of Lexicographical Preference).

   Therefore, $\exists r(x_1) \in \mathbb{Q}$ s.t. $u(x_1, 2) > r(x_1) > u(x_1, 1)$.

   Consider $x_1 > x_1'$. $r(x_1) > u(x_1, 1) > u(x_1', 2) > r(x_1')$. That is, $r(x_1) > r(x_1')$.

   Hence, we have a function $r : \mathbb{R} \to \mathbb{Q}$ that is strictly increasing.

   Thus, $r(\cdot)$ provides a one-to-one mapping from an uncountable set ($\mathbb{R}$) to a countable set $\mathbb{Q}$. This is impossible. □

Remark 1. If X is **finite** and $\succsim$ is a rational preference relation on X, then there is a utility function $u : X \to \mathbb{R}$ that represents $\succsim$.

Proof. See Appendix A. □

1.C. Choice Rules

In reality, the preferences are in Decision Maker (DM)'s mind and we cannot observe them. What we can observe are DM's *choices*. To put the preference theory to work, we need to deduce DM's preferences from her decisions.

A *choice structure* $(\mathscr{B}, C(\cdot))$ consists of two ingredients:

(i) $\mathscr{B}$ is a family (a set) of nonempty subsets of X: that is, every $B \in \mathscr{B}$ is a set $B \subset X$.

- In consumer theory (Chapter 2 & 3), B are budget sets.
- $\mathscr{B}$ does not need to include all possible subsets of X. The convention is to use a fancy capital letter (like $\mathscr{B}$) for a set of sets.

(ii) $C(\cdot)$ is a choice rule that assigns a nonempty subset of chosen elements $C(B) \subset B$ for every $B \in \mathscr{B}$.

- $C(B)$ is a set of *acceptable alternatives*.

Example 1.C.1. $X = \{x, y, z\}, \mathscr{B} = \{\{x, y\}, \{x, y, z\}\}$

Choice Structure 1 $(\mathscr{B}, C_1(\cdot))$: $C_1(\{x, y\}) = \{x\}$, $C_1(\{x, y, z\}) = \{x\}$

Choice Structure 2 $(\mathscr{B}, C_2(\cdot))$: $C_2(\{x, y\}) = \{x\}$, $C_2(\{x, y, z\}) = \{x, y\}$

You might find the choice structure 2 unreasonable. How can the decision maker *not* choose y when the choice set is $\{x, y\}$, but choose y simply when a new item z is added. Consider the following conversation.

Waiter:	Coffee or Tea?
Customer:	Coffee, please.
Waiter:	Sure. Oh sorry, actually we also serve coke. Do you want some coke?
Customer:	*Since coke is available, I'd prefer tea rather than coffee.*

We introduce the following restrictions to eliminate the case that "Since coke is available, I'd prefer tea rather than coffee."

Weak Axiom of Revealed Preference (Reasonable restrictions)

Definition 1.C.1. The choice structure $(\mathscr{B}, C(\cdot))$ satisfies the weak axiom of revealed preference (W.A.R.P) if the following property holds:
If for some $B \in \mathscr{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathscr{B}$ with $x, y \in B'$ and $y \in C(B')$, we must also have $x \in C(B')$.

In the last example, $(\mathscr{B}, C_2(\cdot))$ violates W.A.R.P since $y \in C_2(\{x, y, z\})$, $x, y \in \{x, y\}$, $x \in C_2(\{x, y\})$ but $y \notin C_2(\{x, y\})$.
[Think of $\{x, y, z\}$ as B and $\{x, y\}$ as B' in Definition 1.C.1.]
IDEA: Agent's choice between x and y should not be affected by irrelevant options/alternatives.

Exercise 1.C.1

Consider the choice structure $(\mathscr{B}, C(\cdot))$ with $\mathscr{B} = (\{x, y\}, \{x, y, z\})$ and $C(\{x, y\}) = \{x\}$. Show that if $(\mathscr{B}, C(\cdot))$ satisfies W.A.R.P, then we must have $C(\{x, y, z\}) = \{x\}, = \{z\}$, or $= \{x, z\}$.

Revealed Preference: Preference inferred from/ revealed through Choice

Definition 1.C.2. Given a choice structure $(\mathscr{B}, C(\cdot))$, the revealed preference relation $\succsim^*$ is defined by

$$x \succsim^* y \iff \exists B \in \mathscr{B} \text{ s.t. } x, y \in B \text{ and } x \in C(B).$$

Remark.

1. $x \succsim^* y$ reads "x is revealed at least as good as y".
2. $x \succ^* y$: $\exists B \in \mathscr{B}$ s.t. $x, y \in B$ and $x \in C(B)$, and $y \notin C(B)$. ("x is revealed preferred to y")
3. $\succsim^*$ needs not to be complete or transitive.
4. "Revealed preference" is defined with reference to B, whereas "preference" is defined without reference to B.
5. **Restatement of W.A.R.P**: If $x \succsim^* y$, then $y \not\succ^* x$. (only imposed on $B \in \mathscr{B}$)

Example 1.C.2. Recall Example 1.C.1.

$(\mathscr{B}, C_1(\cdot))$: $x \succ^* y$ and $x \succ^* y$, $x \succ^* z$

$(\mathscr{B}, C_2(\cdot))$: $x \succ^* y$ and $y \succsim^* x \implies$ contradicts W.A.R.P

Useful alternative statement of W.A.R.P $x, y \in B$, $x \in C(B)$, $y \in C(B')$ & $x \notin C(B')$, then $x \notin B'$.

Proof. Proof by contradiction. If $x \in B'$ & $y \in C(B')$, W.A.R.P $\implies$ $x \in C(B')$. □

Exercise 1.C.2

Show that W.A.R.P (Definition 1.C.1) is equivalent to the following property holding: Suppose that $B, B' \in \mathscr{B}$, that $x, y \in B$, and that $x, y \in B'$. Then if $x \in C(B)$ and $y \in C(B')$, we must have $\{x, y\} \subset C(B)$ and $\{x, y\} \subset C(B')$.

1.D. Relationship between Preference Relations & Choice Rules

More precisely, we want to know the relationship between *rational preference* and *W.A.R.P*, the two restrictions we impose on preference and choice rules (revealed preference).

(i) Does Rational Preference imply W.A.R.P? (YES)

(ii) Does W.A.R.P imply Rational Preference? (MAYBE)

Preference Generated Choice Structure Consider rational preference $\succsim$ on X.

Define:

$$C^*(B, \succsim) = \{x \in B : x \succsim y \text{ for every } y \in B\}.$$

- Elements of $C^*(B, \succsim)$ are DM's most preferred alternatives in B.
- Assumption: $C^*(B, \succsim)$ is nonempty for all $B \in \mathscr{B}$.

Exercise 1.D.2

Show that if X is **finite**, then any rational preference relation generates a nonempty choice rule; that is, $C(B) \neq \varnothing$ for any $B \subset X$ with $B \neq \varnothing$. [hint: utilize the result of Remark 1.]

We say that the preference $\succsim$ *generates* the choice structure $(\mathscr{B}, C^*(\cdot, \succsim))$.

Remark. $\succsim$ is defined independently of B. This already hints W.A.R.P is implied.

Proposition 1.D.1. *Suppose $\succsim$ is a rational preference relation. Then the choice structure generated by $\succsim$, $(\mathscr{B}, C^*(\cdot, \succsim))$ satisfies W.A.R.P.*

Proof. Suppose $x, y \in B$ and $x \in C^*(B, \succsim)$.

Also suppose $x, y \in B'$ and $y \in C^*(B', \succsim) \implies y \succsim z, \forall z \in B'$.

Since $x \succsim y, y \succsim z, \forall z \in B' \implies x \succsim z, \forall z \in B' \implies x \in C^*(B', \succsim) \implies$ W.A.R.P is satisfied. ☐

Definition 1.D.1. Given a choice structure $(\mathscr{B}, C(\cdot))$, we say that the rational preference relation $\succsim$ rationalizes $C(\cdot)$ relative to $\mathscr{B}$ if $C(B) = C^*(B, \succsim)$ for all $B \in \mathscr{B}$, that is, if $\succsim$ generates the choice structure $(\mathscr{B}, C(\cdot))$.

Remark.

1. If a rational preference relation rationalizes the choice rule, we can interpret the DM's choices as if she were a preference maximizer.

2. In general, there may be more than one rationalizing preference relation $\succsim$ for a given choice structure $(\mathscr{B}, C(\cdot))$.

   Example. $X = \{x, y\}, \mathscr{B} = \{\{x\}, \{y\}\}, C(\{x\}) = \{x\}, C(\{y\}) = \{y\}$. Any rational preference relation rationalizes $C(\cdot)$.

Example 1.D.1. $X = \{x, y, z\}$, $\mathscr{B} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$[2], $C(\{x, y\}) = \{x\}, C(\{y, z\}) = \{y\}, C(\{x, z\}) = \{z\}$.

This choice structure satisfies the W.A.R.P.

Proof. Use Restatement of W.A.R.P : If $x \succsim^* y$, then $y \not\succ^* x$.

From the choice rules, we know $x \succsim^* y$, $y \succsim^* z$, $z \succsim^* x$. There is no choice rule indicating $y \succ^* x$, $z \succ^* y$, or $x \succ^* z$. Thus, W.A.R.P is not violated. $\square$

However, it cannot be rationalized by a rational preference. Suppose there exists a rational preference $\succsim$ that rationalizes $C(\cdot)$ relative to $\mathscr{B}$, that is, $C^*(B, \succsim) = C(B), \forall B \in \mathscr{B}$. Since $C(\{x, y\}) = \{x\}$, it means $x \succsim y$ & $y \not\succsim x$, i.e. $x \succ y$. Similary, $y \succ x$ & $z \succ x$. Therefore, $\succ$ is not transitive. And thus $\succsim$ cannot be a rational preference.

Remark. W.A.R.P is defined by $\mathscr{B}$. And the choice is not challenged by having to choose from $\{x, y, z\}$.

Exercise 1.D.3

Let $X = \{x, y, z\}$, and consider the choice structure $(\mathscr{B}, C(\cdot))$ with

$$\mathscr{B} = \{\{x, y\}, \{y, z\}, \{x, z\}, \{x, y, z\}\}$$

and $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, and $C(\{x, z\}) = \{z\}$, as in Example 1.D.1. Show that $(\mathscr{B}, C(\cdot))$ must violate W.A.R.P.

[2] $\{x, y, z\}$ is not empirically relevant.

Proposition 1.D.2. *If* $(\mathscr{B}, C(\cdot))$ *is a choice structure such that*

(i) the W.A.R.P is satisfied, [$x \succsim^* y$*, then* $y \not\succ^* x$]

(ii) $\mathscr{B}$ *includes all subsets of* X *of up to three elements,*

then $\exists$ *rational* $\succsim$ *that rationalizes* $C(\cdot)$ *relative to* $\mathscr{B}$*, i.e.,*

$$C(B) = C^*(B, \succsim), \forall B \in \mathscr{B}.$$

Furthermore, this rational preference relation is unique.

Proof. A natural candidate for the rational preference $\succsim$ is $\succsim^*$ (revealed via $(\mathscr{B}, C(\cdot))$).

Step (i) $\succsim^*$ is a rational preference.

Step (ii) $\succsim^*$ rationalizes $(\mathscr{B}, C(\cdot))$, i.e., $C(B) = C^*(B, \succsim^*), \forall B \in \mathscr{B}$.

Step (iii) $\succsim^*$ is unique in rationalizing $(\mathscr{B}, C(\cdot))$.

(i) Rational $\succsim^*$

Transitivity. Suppose $x \succsim^* y$ and $y \succsim^* z$.

Consider $\{x, y, z\} \in \mathscr{B}$. If suffices to prove that $x \in C(\{x, y, z\})$ for $x \succsim^* z$.

$C(\{x, y, z\}) \neq \varnothing$ by assumption.

Suppose $x \in C(\{x, y, z\})$. Then $x \succsim^* z$.

Suppose $y \in C(\{x, y, z\})$. Since $x \succsim^* y$, W.A.R.P implies $x \in C(\{x, y, z\})$, then $x \succsim^* z$.

Suppose $z \in C(\{x, y, z\})$. Since $y \succsim^* z$, W.A.R.P implies $y \in C(\{x, y, z\})$. By the previous case, $x \in C(\{x, y, z\})$, then $x \succsim^* z$.

Completeness. All 2-element subsets belong to $\mathscr{B}$ and $C\{x, y\} \neq \varnothing, \forall x, y \in X \implies x \succsim^* y$ or $y \succsim^* x$.

(ii) $\succsim^*$ rationalizes $C(B), \forall B \in \mathscr{B}$.

Step (a). $C(B) \subset C^*(B, \succsim^*)$

Step (b). $C(B) \supset C^*(B, \succsim^*)$

(a) Suppose $x \in C(B)$. Then $x \succsim^* y, \forall y \in B \iff x \in C^*(B, \succsim^*)$

(b) Suppose $x \in C^*(B, \succsim^*)$. Then $x \succsim^* y, \forall y \in B$.

By definition of $\succsim^*$, $\forall y \in B, \exists B_y \in \mathscr{B}$ (e.g. $B_y = \{x, y\}$) such that $x, y \in B_y$ and $x \in C(B_y)$.

Since $C(B) \neq \varnothing$, either

i. $x \in C(B)$, or

ii. $\exists y \in B \backslash \{x\}$ such that $y \in C(B)$, then by W.A.R.P and $x \succsim^* y$, $x \in C(B)$.

(iii) Uniqueness of $\succsim^*$.

$\mathscr{B}$ includes all 2-element subsets of X. The choice behavoir in $C(\cdot)$ completely pins down whether $x \succsim y$ or $y \succsim x$ for the $\succsim$ which rationalizes $C(\cdot)$. So, $\succsim^*$ is unique. □

Summary of Chapter 1

- A preference relation $\succsim$ is a binary relation on the choice set X.
- $\succsim$ is rational if Completeness & Transitivity.
- Choice function $C(\cdot)$ is defined on $\mathscr{B}$, NOT on X.
- Assumptions on choice structure: W.A.R.P & $C(\cdot) \neq \varnothing$
- Rational Preference implies W.A.R.P.

  When $\mathscr{B}$ includes all 2 & 3-element subsets of X (and $C(B) \neq \emptyset$), then W.A.R.P implies rational preference.

Appendix A

Proof of the claim:

Claim. If X is **finite** and $\succsim$ is a rational preference relation on X, then there is a utility function $u : X \to \mathbb{R}$ that represents $\succsim$.

Proof. Proof by Induction on the number N of the elements of X.

First assume there is no indifference,

1. When $N = 1$, assign any number to the unique element.

2. Let $N > 1$, and suppose the above assertion is true for $N - 1$.

By induction hypothesis, $\succsim$ can be reperesented by utility function $u(\cdot)$ on $\{x_1, ..., x_{N-1}\}$.

It is without loss of generality to assume $u(x_1) > u(x_2) > ... > u(x_{N-1})$. (It is always possible to rearrange items in the set $\{x_1, ..., x_{N-1}\}$ so that $x_{(1)} \succ ... \succ x_{(N-1)}$)

Three exhaustive cases:

Case 1: $x_N \succ x_1$

Case 2: $x_N \prec x_{N-1}$

Case 3: There exists $k \in N$, and $1 < k < N$ such that $x_{k-1} \succ x_N \succ x_k$

We define $\widetilde{u}(\cdot)$ on $\{x_1, ...x_{N-1}, x_N\}$, For $x_i \in \{x_1, ...x_{N-1}\}$, $\widetilde{u}(x_i) = u(x_i)$ and for x_N,

$$\widetilde{u}(x_N) = \begin{cases} u(x_1) + 1 & \text{under Case 1} \\ u(x_{N-1}) - 1 & \text{under Case 2} \\ [u(x_{k-1}) + u(x_k)]/2 & \text{under Case 3} \end{cases}$$

It is easy to check $x \succ y \Leftrightarrow \widetilde{u}(x) > \widetilde{u}(y)$.

Next suppose that there may be indifference in $X = \{x_1, ...x_{N-1}, x_N\}$. For each $n = 1, 2, ..., N$, define $X_n = \{x_n \in X : x_k \sim x_n\}$. Then $\cup_{n=1}^{N} X_n = X$. By transitivity of $\sim$, $X_n \neq X_m$ iff $X_n \cap X_m = \varnothing$. Let M be a subset of $\{1, ..., N\}$ such that $X = \cup_{m \in M} X_m$ and $X_i \neq X_j$ for any $i, j \in M$ with $i \neq j$. Define a relation $\succsim^{\#}$ on $\{X_m : m \in M\}$: $X_i \succsim^{\#} X_j$ iff $x^i \succsim x^j$, where $x^i \in X_i$ and $x^j \in X_j$.

By the definition of M, there is no indifference between two different elements of $\{X_m : m \in M\}$. Thus, by the preceding argument, there exists a utility function $u^{\#}(\cdot)$ representing $\succsim^{\#}$. Define $u : X \to \mathbb{R}$ by $u(x_n) = u^{\#}(X_m)$ if $x_n \in X_m$. Then $u(\cdot)$ represents $\succsim$.

Alternative proof. Consider the indifference directly. Now it becomes $u\left(x_{(1)}\right) \geq u\left(x_{(2)}\right) \geq ... \geq u\left(x_{(N-1)}\right)$ [The ranking may not be unique since for some $k, k \in \mathbb{N}, 1 < k < \mathbb{N}$, $u\left(x_{(k)}\right) = u\left(x_{(k-1)}\right)$.

Three exhaustive cases:

Case 1: $x_N \succsim x_{(1)}$

Case 2: $x_N \precsim x_{(N-1)}$

Case 3: There exists $k \in \mathbb{N}$, and $1 < k < \mathbb{N}$ such that $x_{(k-1)} \succsim x_N \succsim x_{(k)}$

It turns out that the utility function $\widetilde{u}(\cdot)$ defined previously has the property that $x \succsim y \Leftrightarrow \widetilde{u}(x) \geq \widetilde{u}(y)$.

 # Is it feasible to construct utility function without using induction? For example, rank $x_N \succsim ... \succsim x_2 \succsim x_1$ and assign $u(x_i) = i$?

The ranking part is not a formal argument. The induction analysis is nothing but a formalization of this argument. □