Chapter 2. Consumer Choice

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2.A. Introduction

In this chapter, we perform analysis of choice structure in the context of **consumption**. In other words, we analyze consumer demand for commodities.

2.B. Commodities

The decision problem faced by the consumer is to choose the consumption levels of commodities (goods and services).

A *commodity vector* (or commodity *bundle*) is a point

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix} \in \mathbb{R}^L$$

- $\mathbb{R}^L$ is the commodity *space*.

- x_l is the amount of commodity l consumed.

Commodities

Remark. Time (see the example below) and location (see Figure 3), could be built into the definition of a commodity.

For example, x_1 could be bread today, and x_2 could be bread tomorrow. (In this example, we ignore other commodities.) Alice who plans to consume 5 slices of bread today and 6 slices of bread tomorrow would have a commodity vector

$$x = \begin{bmatrix} x_1 = 5 \\ x_2 = 6 \end{bmatrix} \in \mathbb{R}^2.$$

2.C. Consumption Set

The *consumption set* is a subset of the commodity space $\mathbb{R}^L$, denoted by $X \subset \mathbb{R}^L$, whose elements are the consumption bundles that the individual can conceivably consume given the physical and institutional constraints imposed by his environment.

Physical Constraints

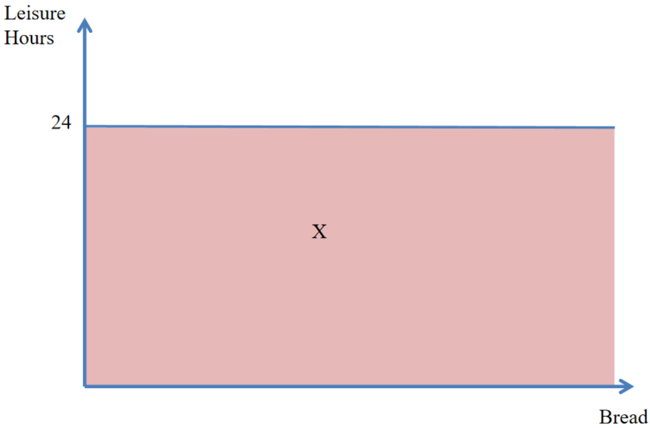

Figure 1: Possible consumption levels of bread and leisure in a day

Physical Constraints

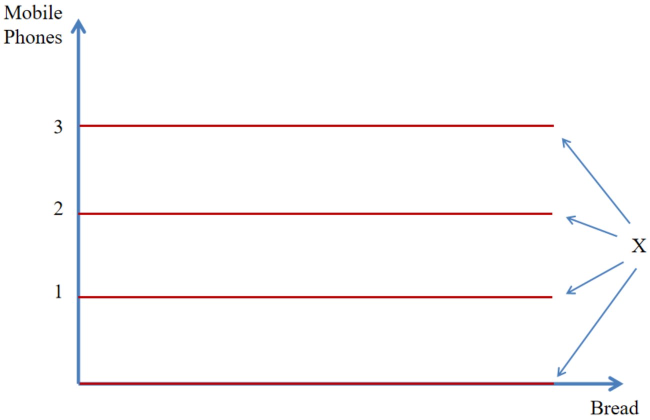

Figure 2: Possible consumption levels of bread and mobile phones

Physical Constraints

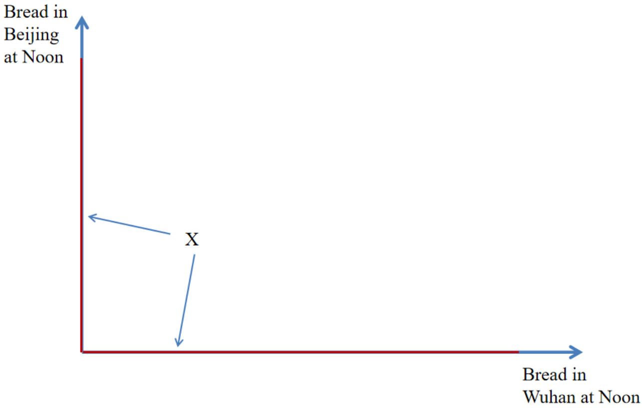

Figure 3: Possible consumption levels of bread in Beijing and Wuhan at noon

Physical Constraints

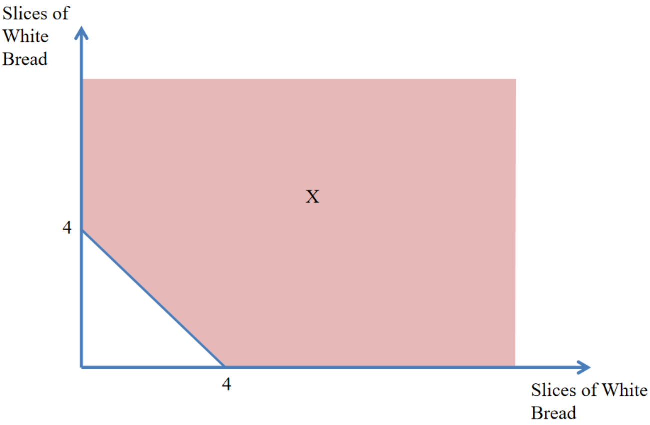

Figure 4: Possible consumption levels of bread where the minimum survival amount is 4 slices and only 2 types of bread are available

There could also be *Institutional* Constraints.

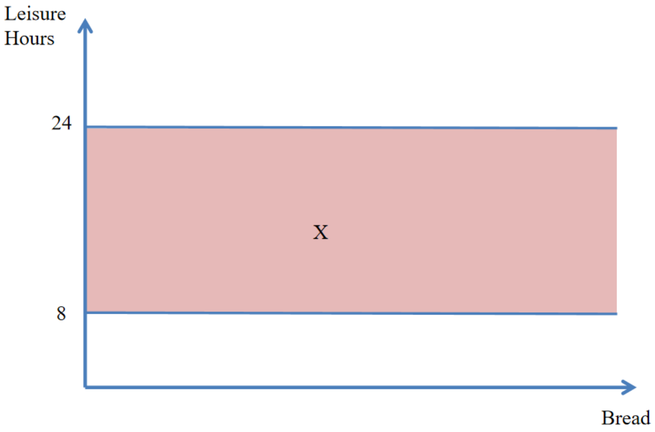

Figure 5: Possible consumption levels of bread and leisure in a day with a law requiring that no one work more than 16 hours a day

Practically, we adopt the simplest consumption set:

$$X = \mathbb{R}^L_+ = \{x \in \mathbb{R}^L : x_l \geq 0 \text{ for } l = 1, 2, ..., L\}.$$

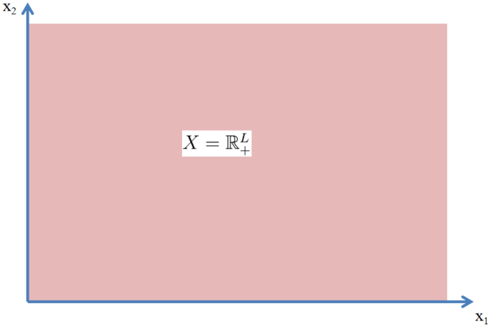

Figure 6: The consumption set $\mathbb{R}^L_+$

Consumption Set

Remark. X is convex: $x \in X, x' \in X \implies \alpha x + (1-\alpha)x' \in X$.

Proof. $x_l \geq 0, x'_l \geq 0, l = 1, ..., L \implies \alpha x_l + (1-\alpha)x'_l \geq 0$

Much of the theory to be developed applies also for more general convex consumption sets (for example, the consumption sets illustrated in Figures 1, 4, 5).[1]

[1] You should check by yourselves that the consumption sets in Figures 1, 4, 5 are convex.

2.D. Competitive Budgets (Affordability)

In addition to the physical and institutional constraints, the consumer also faces *economic* constraint: affordability.

Assumptions:

- L commodities are all traded at public dollar prices.

- Consumers are *price takers*.

Competitive Budgets

Formally, prices are represented by the *price vector*:

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L$$

Assumption. $p \gg 0$, *i.e.*, $p_l > 0, \forall l$.

Competitive Budgets

Question. Do you think this assumption is reasonable?

Competitive Budgets

Counter Examples.

1. Someone invites you: for you, $p_l = 0$.

2. Sometimes parents pay kid to read books: for the kid, $p_l < 0$.

Economic-Affordability Constraint

The affordability of a consumption bundle depends on

1. market prices: $p = (p_1, \cdots, p_L)$

2. consumer's wealth level (in dollars): w

The consumption bundle $x \in \mathbb{R}^L_+$ is affordable if

$$p \cdot x = p_1 x_1 + ... + p_L x_L \leq w.$$

Walrasian budget set

Definition 2.D.1. The Walrasian, or competitive budget set $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ is the set of all feasible consumption bundles for the consumer who faces market prices p and has wealth w.

The consumer's problem is to choose *consumption bundle* x from $B_{p,w}$.

Walrasian budget set

The set $\{x \in \mathbb{R}^L_+ : p \cdot x = w\}$ is called the *budget hyperplane*.

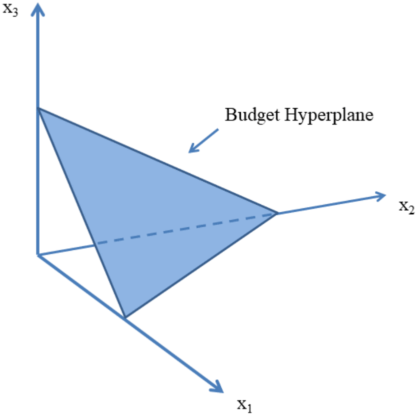

Figure 7: Budget Hyperplane (3 commodities)

Walrasian budget set

When $L = 2$, Budget Hyperplane is Budget Line.

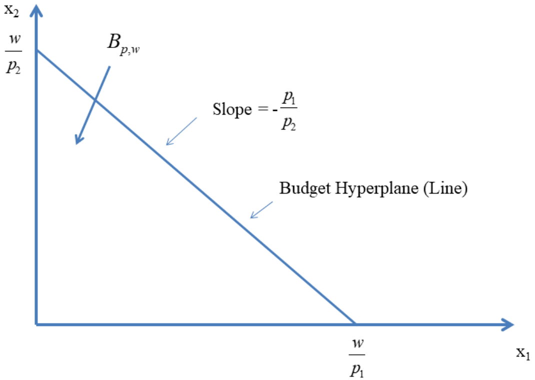

Figure 8: Budget hyperplane (line) for two commodities

Walrasian budget set

The slope $-\frac{p_1}{p_2}$ captures the rate of exchange between the two commodities.

- $\frac{p_1}{p_2}$ describes the units of x_2 the consumer can obtain by giving up one unit of x_1:

  one unit of $x_1 \implies p_1$ of money $\implies \frac{p_1}{p_2}$ units of x_2

Walrasian budget set

p is orthogonal to any vector starting at $\overline{x}$ and lying on the budget hyperplane.

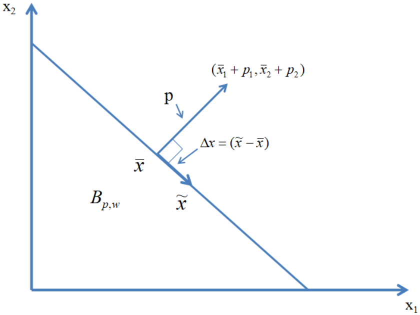

Figure 9: The geometric relationship between p and the budget hyperplane

Walrasian budget set $B_{p,w}$ is convex.

We need to show that for all $x, x' \in B_{p,w}$, $x'' = \alpha x + (1-\alpha)x' \in B_{p,w}$.

Remark. The convexity of $B_{p,w}$ depends on the convexity of the consumption set. $B_{p,w}$ will be convex as long as X is.

2.E. Demand Functions and Comparative Statics

The consumer's *Walrasian* (or *market*, or *ordinary*) *demand correspondence* $x(p, w)$ assigns a set of chosen consumption bundles for each (p, w).

When $x(p, w)$ is single-valued, we refer to it as a *demand function*.

Demand Functions

Assumption.

1. $x(p, w)$ *is homogeneous of degree zero.*

2. $x(p, w)$ *satisfies Walras' law.*

Homogeneous Functions

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is Homogeneous of Degree k if for any $\alpha > 0$,

$$f(\alpha x_1, \alpha x_2, ..., \alpha x_n) = \alpha^k f(x_1, x_2, ..., x_n).$$

Examples of Homogeneous Functions

1. $f(x, y) = xy$ is Homogeneous of Degree 2.
2. $f(x, y, z) = \frac{x}{y} + \frac{2z}{x}$ is Homogeneous of Degree 0.
3. $f\left(x_1, x_2\right) = Ax_1^a x_2^b$ is Homogeneous of Degree $a + b$.
4. $f\left(x_1, x_2\right) = x_1 + x_2^2$ is not a Homogeneous Function.

Homogeneous of Degree Zero

Definition 2.E.1. The Walrasian demand correspondence $x(p, w)$ is homogeneous of degree zero (H.D.$\emptyset$) if $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and $\alpha > 0$.

Remark. Since $B_{p,w} = B_{\alpha p, \alpha w}$, H.D.$\emptyset$ means that individual's choice depends only on the set of feasible points.

Remark. Implication of H.D.$\emptyset$: it is without loss to *normalize* the level of one of the $L+1$ independent variables at an arbitrary level.

Walras' Law

Definition 2.E.2. The Walrasian demand correspondence $x(p, w)$ satisfies Walras' law if for every $p \gg 0$ and $w > 0$, we have $p \cdot x = w$ for all $x \in x(p, w)$.

Remark. Walras' law says that the consumer fully expends his wealth.

Walras' Law

Question. Is Walras' law reasonable?

Walras' Law

Question. Is Walras' law reasonable?

It's more reasonable if w refers the life-time income and x refers to life-time demands. Even then, it's still controversial.

Demand Functions

For the remainder of the section, we assume that $x(p, w)$ is single-valued, continuous, and differentiable.

$\mathbf{x}(\mathbf{p}, \mathbf{w})$ and Choice-base Approach (in Chapter 1)

Recall that a choice structure $(\mathscr{B}, C(\cdot))$ consists of two ingredients:

(i) $\mathscr{B}$ is a family of nonempty subsets of X. Every $B \in \mathscr{B}$ is a budget set.

(ii) $C(\cdot)$ is a choice rule. It maps every set $B \in \mathscr{B}$ to a nonempty set $C(B) \subset B$.

$\mathbf{x(p, w)}$ and Choice-base Approach (in Chapter 1)

The family of Walrasian budget sets is

$$\mathscr{B}^{\mathscr{W}} = \{B_{p,w} : p \gg 0, w > 0\}.$$

Remark. $\mathscr{B}^{\mathscr{W}}$ does not include all possible subsets of X.

Since the price-wealth pair (p, w) determines the Walrasian budget set $B_{p,w}$ faced by consumer, we have

$$C(B_{p,w}) = x(p, w).$$

Hence, $(\mathscr{B}^{\mathscr{W}}, x(p, w))$ is a choice structure.

Comparative statics (with respect to p and w)

The examination of a change in outcome in response to a change in underlying economic parameters is known as *comparative statics* analysis.

This section examines how the consumer's choice would vary with changes in his wealth and in prices.

Wealth Effects

For fixed prices $\overline{p}$, $x(\overline{p}, w)$ is called the consumer's *Engel function*. Its image in $\mathbb{R}^L_+$, $E_{\overline{p}} = \{x(\overline{p}, w) : w > 0\}$ is the *wealth expansion path*.

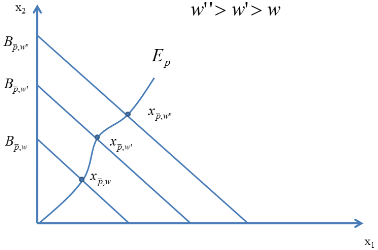

Figure 10: Wealth expansion path at $\overline{p}$

Wealth Effects

The derivative $\frac{\partial x_l(p,w)}{\partial w}$ is the *wealth effect* for the l^{th} good.

- A commodity l is *normal* at (p, w) if $\frac{\partial x_l(p,w)}{\partial w} \geq 0$.

- A commodity l is *inferior* at (p, w) if $\frac{\partial x_l(p,w)}{\partial w} < 0$.

In matrix notation, the wealth effects are

$$D_w x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \vdots \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix} \in \mathbb{R}^L.$$

Price Effects

The demand function for good l could be represented as a function of p_l, keeping other things equal, i.e., $x(p_l, \overline{p}_{-l}, \overline{w})$.

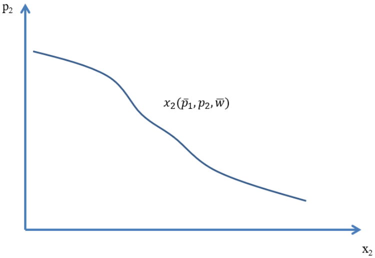

Figure 11: Demand for good 2 as a function of its price

Price Effects

Another useful representation of the consumers' demand at different prices p_l is the locus of points demanded in $\mathbb{R}^L_+$, for fixed p_{-l} and w. This is known as an *offer curve*.

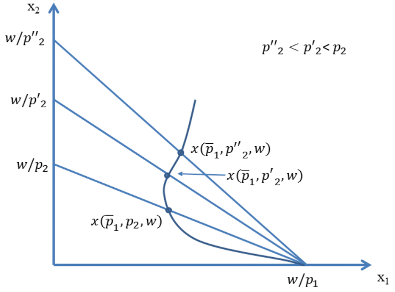

Figure 12: Offer Curve

Price Effects

The derivative $\frac{\partial x_l(p,w)}{\partial p_k}$ is the *price effect* of p_k on the demand for good l.

- Good l is a *Giffen good* if $\frac{\partial x_l(p,w)}{\partial p_l} > 0$. (Example: potatoes at low wealth level)

In matrix notation, the price effects are

$$D_p x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \dots & \frac{\partial x_1(p,w)}{\partial p_L} \\ & \ddots & \\ \frac{\partial x_L(p,w)}{\partial p_1} & \dots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}.$$

Implications of homogeneity for price and wealth effects

Proposition 2.E.1. *If the Walrasian demand function* $x(p, w)$ *is homogeneous of degree zero, then for all* p *and* w*:*

$$\sum_{k=1}^{L} \frac{\partial x_l(p, w)}{\partial p_k} p_k + \frac{\partial x_l(p, w)}{\partial w} w = 0, \text{ for } l = 1, ..., L. \quad \text{(2.E.1)}$$

In matrix notation, it is expressed as

$$D_p x(p, w) p + D_w x(p, w) w = 0. \quad \text{(2.E.2)}$$

Implication of homogeneity for price and wealth effects

Divide the expression by x_l:

$$\sum_{k=1}^{L} \frac{\partial x_l(p,w)}{\partial p_k} \frac{p_k}{x_l(p,w)} + \frac{\partial x_l(p,w)}{\partial w} \frac{w}{x_l(p,w)} = 0, \text{ for } l = 1, ..., L.$$

i.e.,

$$\sum_{k=1}^{L} \varepsilon_{lk}(p,w) + \varepsilon_{lw}(p,w) = 0, \text{ for } l = 1, ..., L. \qquad (2.E.3)$$

Intuition: The above equation describes the precentage change in x_l if all prices and wealth changes 1%. Basically, the equation captures the definition of H.D.$\emptyset$.

Implications of Walras' Law for price and wealth effects

Proposition 2.E.2. *If the Walrasian demand function* $x(p,w)$ *satisfies the Walras' Law, then for all* p *and* w *:*

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) = 0, \text{ for } k = 1,2,...,L, \quad (2.E.4)$$

or written in matrix notation,

$$p \cdot D_p x(p,w) + x(p,w)^T = 0^T. \quad (2.E.5)$$

Intuition: Total expenditure cannot change in response to a change in prices.

Implications of Walras' Law for price and wealth effects

Proposition 2.E.3. *If the Walrasian demand function $x(p, w)$ satisfies Walras' Law, then for ALL p and w:*

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p, w)}{\partial w} = 1, \qquad (2.E.6)$$

or, written in matrix natation,

$$p \cdot D_w x(p, w) = 1. \qquad (2.E.7)$$

Intuition: Total expenditure must change by an amount equal to any wealth change.

2.F. Weak Axiom of Revealed Preference and Law of Demand

Implicit assumptions: $x(p, w)$ is single-valued, homogeneous of degree zero, and satisfies Walras' Law.

W.A.R.P and Law of Demand

Definition 2.F.1. The Walrasian demand function $x(p, w)$ satisfies the weak axiom of revealed preference (W.A.R.P) if the following holds for any two price-wealth situations (p, w) and (p', w'): If $p \cdot x(p', w') \leq w$ and $x(p', w') \neq x(p, w)$,[2] then $p' \cdot x(p, w) > w'$.

[2]Note that $x(p, w)$ is the demand given (p, w) and $x(p', w')$ is the demand given (p', w').

Definition stated using language in Chapter 1

Let $B_{p,w}$ denote the budget set given p and w; and $B_{p',w'}$ denote the budget set given p' and w'. $p \cdot x(p', w') \leq w$ means that $x(p', w')$ is also affordable under $B_{p,w}$. Through the choice given $B_{p,w}$, $x(p, w)$ is revealed preferred to $x(p', w')$. Therefore, by W.A.R.P, it must not be revealed that $x(p', w')$ is preferred to $x(p, w)$. In other words, if $x(p, w)$ is not chosen given the budget $B_{p',w'}$, it must be that it is not affordable, i.e., $p' \cdot x(p, w) > w'$, or $x(p, w) \notin B_{p',w'}$.

Demand Satisfying W.A.R.P

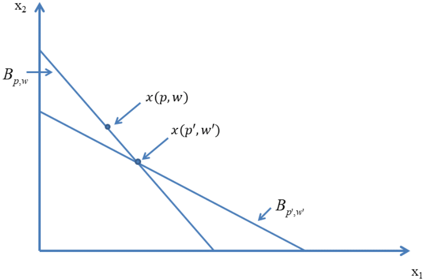

Figure 13: Demand satisfying W.A.R.P

Violation of W.A.R.P

W.A.R.P may be violated only if both $x(p, w)$ and $x(p', w')$ belong to both $B_{p,w}$ and $B_{p',w'}$.

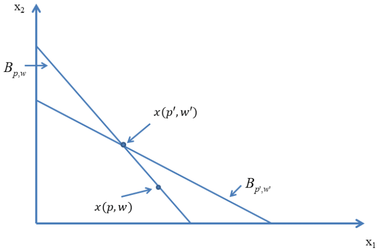

Figure 14: Demand violating W.A.R.P

Implications of W.A.R.P

Uncompensated price change

An uncompensated price change is a change in price without a corresponding change in wealth.

Such a price change would affect the consumer in two ways:

- change the relative cost of commodities;
- change the consumer's real wealth.

W.A.R.P and Uncompensated price change

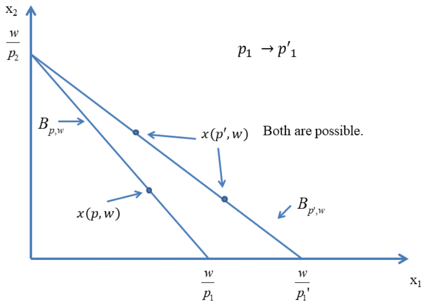

Figure 15: Uncompensated price change

Assuming W.A.R.P, no prediction on change in demand can be drawn.

Compensated price change

Imagine a situation in which a change in prices is accompanied by a change in the consumer's wealth that makes her initial consumption bundle just affordable at the new prices. That is, $w' = p' \cdot x(p, w)$. The wealth adjustment is $\Delta w = \Delta p \cdot x(p, w)$. This kind of wealth adjustment is called *Slutsky wealth compensation*. The price changes that are accompanied by compensating wealth changes are called *(Slutsky) compensated price changes*.

W.A.R.P and Compensated price change

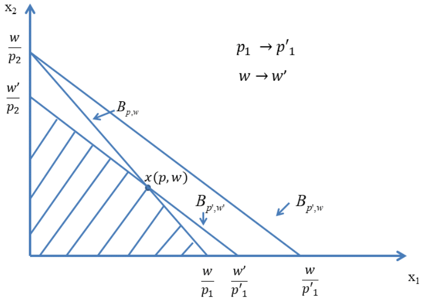

Figure 16: Compensated price change

- x_1 *must increase after the decrease of* p_1 *and an associated wealth compensation.*
- This is the *Compensated Law of Demand*.

W.A.R.P and Compensated law of demand

In Proposition 2.F.1, we will define *Compensated Law of Demand* and formally show that W.A.R.P implies Compensated Law of Demand.

Furthermore, we will prove that the converse is also true: Compensated Law of Demand implies W.A.R.P.

W.A.R.P

Next, we present a useful lemma which makes it easier to check whether a demand function satisfies W.A.R.P (for all price-wealth changes).

Lemma 1. *W.A.R.P holds for all price-wealth changes if and only if it holds for all compensated price changes.*

W.A.R.P and Compensated law of demand

Proposition 2.F.1. *Suppose that the Walrasian demand function $x(p,w)$ is homogeneous of degree zero and satisfies Walras' Law, Then $x(p,w)$ satisfies W.A.R.P if and only if $x(p,w)$ satisfies Compensated Law of Demand, that is, for ANY compensated price change from an initial situation (p,w) to a new price-wealth pair $(p',w') = (p', p' \cdot x(p,w))$, we have*

$$(p'-p) \cdot [x(p',w') - x(p,w)] \leq 0, \qquad \text{(2.F.1)}$$

with strict inequality whenever $x(p,w) \neq x(p',w')$.

W.A.R.P and Compensated Law of Demand

Remark. The inequality (2.F.1) is interpreted as *Compensated Law of Demand* since

- demand and price move in opposite directions (law of demand), and
- it only holds for compensated price changes.

W.A.R.P and Compensated Law of Demand

Remark. As illustrated in Figure 15, W.A.R.P does not generate definitive prediction on the demand changes in response to *uncompensated* price changes.

W.A.R.P and Differentiable Demand

Consider a differentiable change in price dp, compensated by the change in wealth

$$dw = x(p, w) \cdot dp.$$

By chain rule,

$$dx = \left[D_p x(p, w) + D_w x(p, w) x(p, w)^T\right] dp \qquad \text{(2.F.8)}$$

Slutsky Matrix

Define

$$S(p, w) = D_p x(p, w) + D_w x(p, w) x(p, w)^T$$

as the *substitution matrix* or *Slutsky matrix*.

Slutsky Matrix

In matrix notation, it is

$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ & \ddots & \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix},$$

where the $(l,k)^{th}$ entry is

$$s_{l,k}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w).$$

$s_{l,k}(p,w)$ are known as *substitution effects*.

Slutsky Matrix

$s_{l,k}(p, w)$ is the change in demand for good l given a change in p_k and a compensating change in w.

Negative semidefiniteness of Slutsky matrix

Proposition 2.F.2. *If a differentiable Walrasian demand function $x(p, w)$ satisfies Walras' Law, homogeneous of degree zero, and W.A.R.P, then at any (p, w), the Slutsky matrix $S(p, w)$ satisfies $v \cdot S(p, w)v \leq 0$ for any $v \in \mathbb{R}^L$. i.e. $S(p, w)$ is negative semidefinite.*

Proposition 2.F.1 implies

$$dp \cdot dx \leq 0.$$

Together with Equation (2.F.8) gives the result.

Slutsky Matrix

Remark. Proposition 2.F.2 does not imply, in general, that the matrix $S(p, w)$ is symmetric.

- For $L = 2$, $S(p, w)$ is necessarily symmetric.

  (Exercise 2.F.11)

- When $L > 2$, $S(p, w)$ is not necessarily symmetric, under the assumptions so far.

- Symmetry of $S(p, w)$ is connected with maximization of rational preferences. (will be introduced in Chapter 3)

Slutsky Matrix

Corollary. *The substitution effect of good l with respect to its own price is always nonpositive, i.e.,* $s_{ll}(p, w) \leq 0$.

Remark. An implication of $s_{ll}(p, w) \leq 0$ is that a good can be a Giffen good at (p, w) only if it is inferior.

Slutsky Matrix

Remark. H.D.$\emptyset$ + Walras' law + Negative semidefiniteness of $S(p,w) \not\Longrightarrow$ W.A.R.P.

Compare with Proposition 2.F.2:

H.D.$\emptyset$ + Walras' law + W.A.R.P $\Longrightarrow$ Negative semidefiniteness of $S(p,w)$

Slutsky Matrix

Example. Consider $L = 3$ and $X = \mathbb{R}$ and $x(p, w)$ is

$$x_1(p, w) = \frac{p_2}{p_3} \quad \& \quad x_2(p, w) = -\frac{p_1}{p_3} \quad \& \quad x_3(p, w) = \frac{w}{p_3}.$$

(a) $x(p, w)$ is H.D.$\emptyset$ and satisfies Walras' law.

(b) $x(p, w)$ violates W.A.R.P.

(c) $v \cdot S(p, w)v = 0$ for all $v \in \mathbb{R}^3$.

Slutsky Matrix

Remark. H.D.$\emptyset$ + Walras' law + $v \cdot S(p, w)v < 0$ whenever $v \neq \alpha p$ for any scalar $\alpha \implies$ W.A.R.P.

More properties on Slutsky matrix

Proposition 2.F.3. *Suppose that the Walrasian demand function $x(p, w)$ is differentiable, homogeneous of degree zero, and satisfies Walras' law. Then, $p \cdot S(p, w) = 0$ and $S(p, w)p = 0$ for any (p, w).*

More properties on Slutsky matrix

It follows from Proposition 2.F.3 that the negative semidefiniteness of $S(p, w)$ established in Proposition 2.F.2 cannot be extended to negative definiteness.

As an example, see Exercise 2.F.17.

Choice-based Approach and Preference-based Approach

Remark. $\mathscr{B}^{\mathscr{W}} = \{B_{p,w} : p \gg 0, w > 0\}$ does not include every possible budget; in particular, it does not contain all two- and three-element subsets of X.

Therefore, choice-based approach $\neq$ preference-based approach.

Example 2.F.1. In a three-commodity world, consider the three budget sets determined by the price vectors $p^1 = (2,1,2)$, $p^2 = (2,2,1)$, $p^3 = (1,2,2)$ and wealth $= 8$ (the same for the three budgets). Suppose that the respective (unique) choices are $x^1 = (1,2,2)$, $x^2 = (2,1,2)$, $x^3 = (2,2,1)$. For these three budgets, any two pairs of choices satisfy W.A.R.P but x^3 is revealed preferred to x^2, x^2 is revealed preferred to x^1, and x^1 is revealed preferred to x^3.

Summary of Chapter 2

Taking choice as the primative, we look at the implications of these assumptions:

(i) $x(p, w)$ is homogeneous of degree zero.

(ii) $x(p, w)$ satisfies Walras' Law.

(iii) $x(p, w)$ satisfies the W.A.R.P $\iff$ Compensated Law of Demand

(iv) $x(p, w)$ is also differentiable $\implies$ Slutsky matrix is negative semidefinite.