Chapter 5. Production

5.A. Introduction

In this chapter, we study the **supply side** of the economy. In particular, we study how goods and services are produced by "firms". Here, we view firms as "black boxes", transforming inputs into outputs. That is, we ignore the organizational structure within the firms. Please note that this simplification is for the purpose of analyzing market behavior. The study of organizational structure, witch falls outside of the scope of this chapter, is also important and interesting.

5.B. Production Sets

- We consider an economy with L commodities.
- *Production vector* (**including both inputs & outputs**) $y = (y_1, ..., y_L) \in \mathbb{R}^L$ describes the (net) outputs.
  - If $y_l > 0$, l is an output;
  - If $y_l \leq 0$, l is an input.

  Example 5.B.1. Suppose that $L = 5$, Then $y = (-5, 2, -6, 3, 0)$ means that

  (a) 2 and 3 units of Good 2 and 4 are produced;

  (b) 5 and 6 units of Good 1 and 3 are used;

  (c) Good 5 is neither produced or used.

- The set of all production vectors that constitute technologically feasible plans is called the *production set* $Y \subset \mathbb{R}^L$.
  - Any $y \in Y$ is feasible;
  - Any $y \notin Y$ is not feasible.
- We can describe the production set Y by a *transformation function* $F(\cdot)$.
  - The production set $Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$.
  - $\{y \in \mathbb{R}^L : F(y) = 0\}$ is called the *transformation frontier*.

- Figure 1 below presents the production function and transformation frontier for two goods.

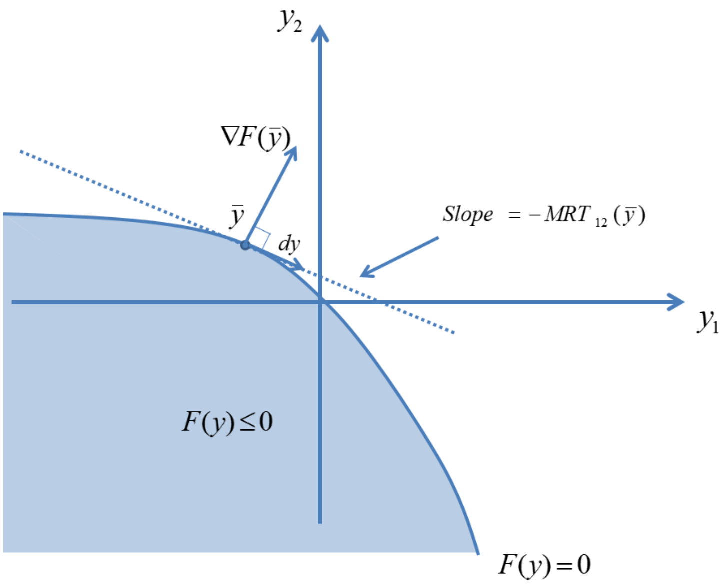

Figure 1: Production Function and Transformation Frontier

- Consider changes in y while staying on $F(y) = 0$. For such changes dy along the frontier, we have $dy \cdot \nabla F(y) = 0$.

- Suppose only y_l & y_k change.

$$dF(\bar{y}) = \frac{\partial F(\bar{y})}{\partial y_l} dy_l + \frac{\partial F(\bar{y})}{\partial y_k} dy_k = 0$$
$$\iff \frac{dy_k}{dy_l} = -\frac{\partial F(\bar{y})/\partial y_l}{\partial F(\bar{y})/\partial y_k} = -MRT_{lk}(\bar{y}).$$

$MRT_{lk}(\bar{y})$ is called the *marginal rate of transformation (MRT) of good l for good k at $\bar{y}$.*

Technologies with Distinct Inputs and Outputs

- Suppose there are M outputs and $L - M$ inputs.
  - let $q = (q_1, ..., q_M) \geq 0$ denote the outputs.
  - let $z = (z_1, ..., z_{L-M}) \geq 0$ denote the inputs.
  - e.g. $(y_{L-M+1}, ..., y_L) = (q_1, ..., q_M)$; $(y_1, ..., y_{L-M}) = -(z_1, ..., z_{L-M})$.

- Single-output technology
  - *Production function*: $f(z)$, where $z = (z_1, ..., z_{L-1}) \geq 0$
  - Output: $q \leq f(z)$
  - Production set:

$$Y = \{(-z_1, ..., -z_{L-1}, q) : q - f(z_1, ..., z_{L-1}) \leq 0 \text{ and } (z_1, ..., z_{L-1}) \geq 0\}$$

- Holding the level of output fixed, we define *Marginal Rate of Technological Substitution (MRTS) of input l for input k at $\bar{z}$* as follows:

$$MRTS_{lk}(\bar{z}) = \frac{\partial f(\bar{z})/\partial z_l}{\partial f(\bar{z})/\partial z_k}$$

  - $MRTS_{lk}(\bar{z})$ is the same as $MRT_{lk}(\bar{z}, \bar{q})$, simply a renaming for the substitution between inputs in a single-output case.

Example 5.B.2. Cobb-Douglas Production Function:

$$f(z_1, z_2) = z_1^\alpha z_2^\beta, \text{ where } a \geq 0, \beta \geq 0. \,(f(z_1, z_2): \text{ output}, z_1: \text{ input } 1, z_2: \text{ input } 2.)$$

MRTS at $z = (z_1, z_2)$ is

$$MRTS_{12}(z) = \frac{\partial f(z_1, z_2)/\partial z_1}{\partial f(z_1, z_2)/\partial z_2} = \frac{\alpha z_1^{\alpha-1} z_2^\beta}{\beta z_1^\alpha z_2^{\beta-1}} = \frac{\alpha z_2}{\beta z_1}.$$

Remark. In percentage change terms

$$\left[\frac{\partial f(z_1, z_2)}{\partial z_1} \frac{z_1}{f(z_1, z_2)}\right] \Big/ \left[\frac{\partial f(z_1, z_2)}{\partial z_2} \frac{z_2}{f(z_1, z_2)}\right] = \frac{\alpha z_2}{\beta z_1} \frac{z_1}{z_2} = \frac{\alpha}{\beta}.$$

Commonly Assumed Properties of Production Sets

(i) Y is nonempty.

(ii) Y is closed. (technical)

(iii) No free lunch: If $y \geq 0$, then $y = 0$. The idea is that no commodities can be created out of thin air. Production of any commodity requires consumption of some other commodities.

(iv) Possibility of inaction: $0 \in Y$.

(v) Free disposal: If $y \in Y$ and $y' \leq y$, then $y' \in Y$.

- Extra amount of inputs (or outputs) can be disposed at no cost.

(vi) Irreversibility: Suppose $y \in Y$ and $y \neq 0$, then $-y \notin Y$.

- For example, One cannot effortlessly disassemble an iPad and turn it back into its original parts in perfect condition.
- Figure 2 and Figure 3 below depict reversible and irreversible technology respectively.

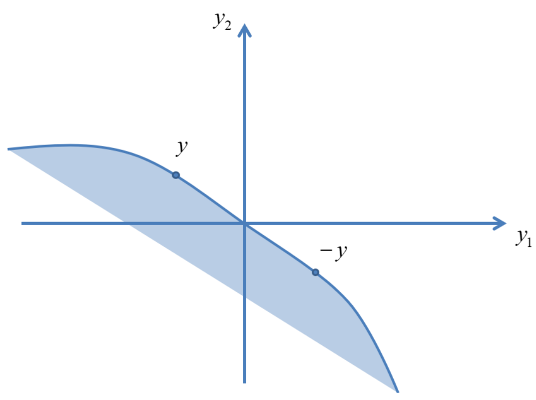

Figure 2: Reversible Technology

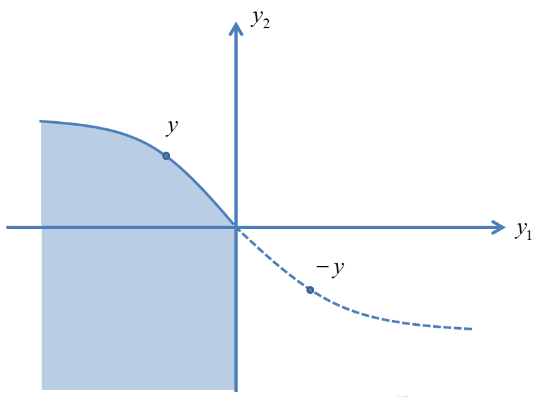

Figure 3: Irreversible Technology

(vii) Nonincreasing returns to scale: $y \in Y$ and $\alpha \in [0, 1] \implies \alpha y \in Y$.

- Smaller scale is more efficient: Half the inputs will get you more than half the outputs.

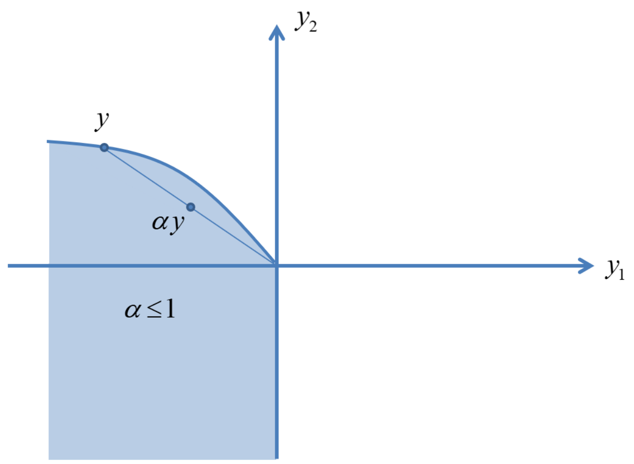

Figure 4: Nonincreasing Returns to Scale Technology

(viii) Nondecreasing returns to scale: $y \in Y$ and $\alpha \geq 1 \implies \alpha y \in Y$.

- Larger scale is more efficient. Double the inputs will get you more than double the outputs.

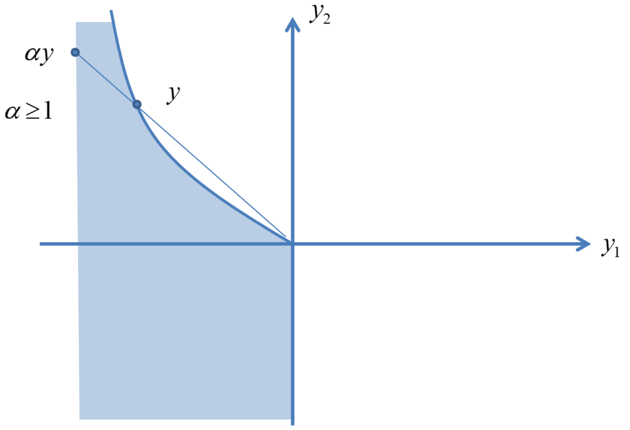

Figure 5: Nondecreasing Returns to Scale Technology

(ix) Constant returns to scale (Cone): $y \in Y$ and $\alpha \geq 0 \implies \alpha y \in Y$.

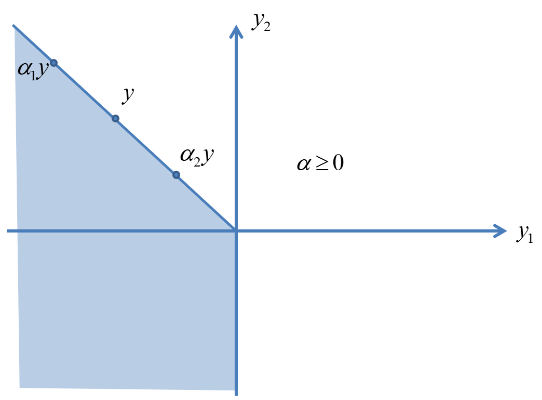

Figure 6: CRS (2 commodities)

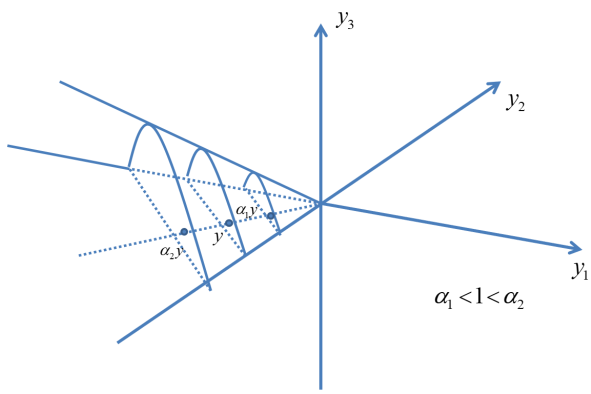

Figure 7: CRS (3 commodities)

Exercise 5.B.2

Suppose that $f(\cdot)$ is the production function associated with a single-output technology, and let Y be the production set of this technology. Show that Y satisfies constant returns to scale if and only if $f(\cdot)$ is homogeneous of degree one.

(x) Additivity: Suppose $y \in Y$ and $y' \in Y$. Then $y + y' \in Y$.

- Alternatively, $Y + Y \subset Y$.
- If $y \in Y$, then $ky \in Y$ for all $k \in \mathbb{Z}$.
- This captures an economy with **free entry**: Any existing technology can be added to the existing technologies.

(xi) Convexity: $y, y' \in Y$ and $\alpha \in [0,1] \implies \alpha y + (1-\alpha)y' \in Y$.

- Convexity implies nonincreasing returns to scale: if inaction is possible (i.e., $0 \in Y$), then convexity implies that for any $\alpha \in [0,1]$, $\alpha y = \alpha y + (1-\alpha)0 \in Y$.
- "Balanced" inputs (outputs) are weakly more productive (less costly) than "unbalanced" ones.

Exercise 5.B.3

Show that for a single-output technology, Y is convex if and only if the production function $f(\cdot)$ is concave.

(xii) Convex cone: Y is a convex cone if for any production vector $y, y' \in Y$ and constants $\alpha \geq 0$ & $\beta \geq 0$, we have $\alpha y + \beta y' \in Y$.

- Note that $\alpha y + \beta y'$ can be written as $\gamma[\theta y + (1-\theta)y']$ for some $\gamma \geq 0$ and $\theta \in [0,1]$.[1] The convex combination between y and y' captures the *convex* part of the definition, and $\gamma \geq 0$ captures the *cone* part of the definition.
- The production sets depicted in Figure 6 and Figure 7 are both convex cones.

Proposition 5.B.1. *The production set Y is additive and satisfies the nonincreasing returns condition if and only if it is a convex cone.*

Proof.

1. "$\Longleftarrow$" part: Suppose Y is a convex cone. That is, for any $y, y' \in Y$ and $\alpha \geq 0$ & $\beta \geq 0$, we have $\alpha y + \beta y' \in Y$.

[1] More specifically, we could let $\gamma = \alpha + \beta$ and $\theta = \frac{\alpha}{\alpha+\beta}$, we have $(\alpha+\beta)[\frac{\alpha}{\alpha+\beta}y + \frac{\beta}{\alpha+\beta}y'] \in Y$.

a) Let $\alpha = \beta = 1$. Then, for any y, $y' \in Y$, we have $y + y' \in Y$. Therefore, Y is additive.

b) Let $\beta = 0$ and $\alpha \in [0, 1]$. Then, for any $y \in Y$ and $\alpha \in [0, 1]$, we have $\alpha y \in Y$. Therefore, Y is nonincreasing returns to scale.

2. "$\Longrightarrow$" part: Consider any $y, y' \in Y$ and $\alpha \geq 0, \beta \geq 0$.

- Let $k \in \mathbb{Z}$ such that $k > \max\{\alpha, \beta\}$. Then by additivity, $ky \in Y$ and $ky' \in Y$.
- Since $\frac{\alpha}{k} < 1$, by nonincreasing returns to scale, $\frac{\alpha}{k} ky \in Y$. That is, $\alpha y \in Y$. Similarly, $\beta y' \in Y$.
- By additivity, $\alpha y + \beta y' \in Y$.
- Since for any $y, y' \in Y$ and $\alpha \geq 0$ & $\beta \geq 0$, $\alpha y + \beta y' \in Y$, Y is a convex cone. $\square$

Proposition 5.B.2. *For any convex production set $Y \subset \mathbb{R}^L$ with $0 \in Y$, there is a constant returns, convex production set $Y' \subset \mathbb{R}^{L+1}$ s.t. $Y = \{y \in \mathbb{R}^L : (y, -1) \in Y'\}$.*

Remark. Here "there exists" only means that there exists such an interpretation. It doesn't really mean that the technology necessarily exists.

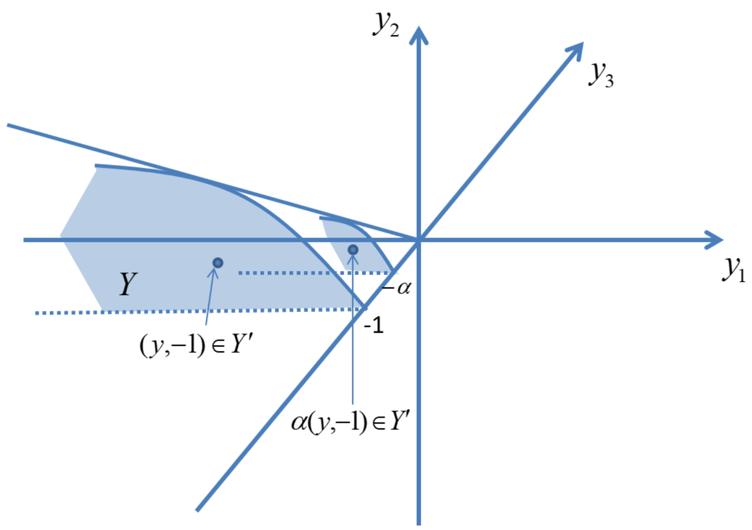

Figure 8: Constant Returns Production Set

Proof. Let $Y' = \{y' \in \mathbb{R}^{L+1} : y' = \alpha(y, -1)$ for some $y \in Y$ & $\alpha \geq 0\}$. See Figure 8. We now check that Y' is constant returns and convex.

- Constant returns: We need to show that if $y' \in Y'$, then $\gamma y' \in Y'$ for all $\gamma \geq 0$.

  $y' \in Y'$ means $y' = \alpha(y, -1)$ where $y \in Y$ and $\alpha \geq 0$, then for any $\gamma \geq 0$, $\gamma y' = \gamma\alpha(y, -1) \in Y'$ since $y \in Y$ and $\gamma\alpha \geq 0$.

- Convexity: We need to show that if $y_1', y_2' \in Y', \beta \in [0,1]$, then $\beta y_1' + (1-\beta)y_2' \in Y'$.

  $y_1' \in Y'$ and $y_2' \in Y'$ mean $y_1' = \alpha_1(y_1, -1)$ where $y_1 \in Y$ and $\alpha_1 \geq 0$ and $y_2' = \alpha_2(y_2, -1)$ where $y_2 \in Y$ and $\alpha_2 \geq 0$, then for any $\beta \in [0,1]$,

$$\begin{aligned}
&\beta y_1' + (1-\beta)y_2' = \beta\alpha_1(y_1, -1) + (1-\beta)\alpha_2(y_2, -1) \\
=&[\alpha_1\beta + \alpha_2(1-\beta)]\left(\frac{\alpha_1\beta}{\alpha_1\beta + \alpha_2(1-\beta)}y_1 + \frac{\alpha_2(1-\beta)}{\alpha_1\beta + \alpha_2(1-\beta)}y_2, -1\right) \in Y'
\end{aligned}$$

  since $\frac{\alpha_1\beta}{\alpha_1\beta+\alpha_2(1-\beta)}y_1 + \frac{\alpha_2(1-\beta)}{\alpha_1\beta+\alpha_2(1-\beta)}y_2 \in Y$ ($\because Y$ is convex) and $\alpha_1\beta + \alpha_2(1-\beta) \geq 0$. ☐

Remark. Y is not constant returns to scale. But it can be the cross-section of a constant returns to scale $Y' \subset \mathbb{R}^{L+1}$. In essence, the implication is that in a competitive, convex setting, there may be little loss of conceptual generality in limiting to constant returns technologies.

Remark (on the concept of production set). Production set is a description of technology. So, if inputs (including good "$L+1$") are there, production should be scalable. In other words, decreasing returns to scale observed in real life must be a reflection of scarcity of inputs.

5.C. Profit Maximization and Cost Minimization

- L commodities, priced at $p = (p_1, ...p_L) \gg 0$.
- Firm is *price-taking.*
- Firm's objective is to maximize profit.
- Assume *(i) nonemptiness*, *(ii) closedness*, and *(v) free disposal.*

Profit Maximization Problem

- Profit $= p \cdot y$. This is because y includes both inputs (as negative) and outputs (as positive). Besides, profit can come from multiple products.
- Profit maximization problem

$$\max_{y \in \mathbb{R}^L} p \cdot y$$

$$\text{s.t. } y \in Y(\text{ or } F(y) \leq 0)$$

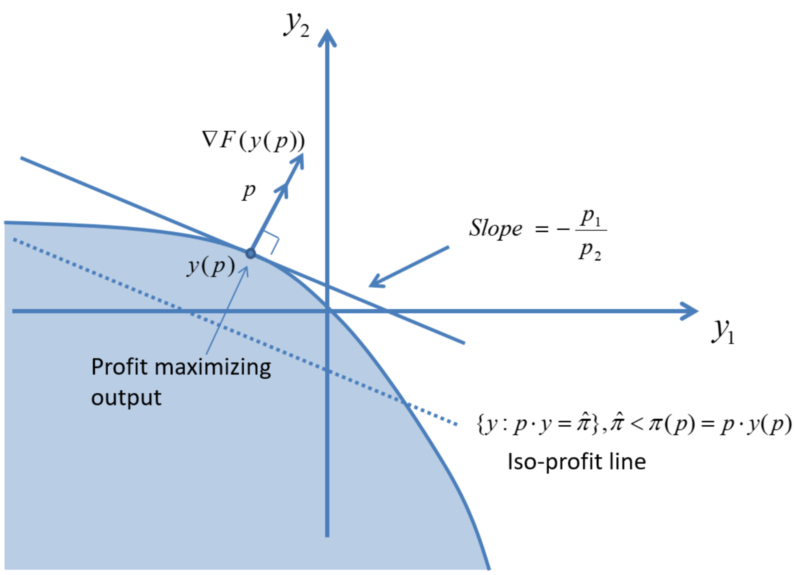

Figure 9: Profit Maximization Problem

- In general, the profit maximization problem may have multiple solutions ($y(p)$ is a set rather than s single vector), or no solution.
- Lagrange Function:

$$\mathcal{L} = p \cdot y - \lambda F(y)$$

- Kuhn-Tucker Conditions:[2]

$$\frac{\partial \mathcal{L}}{\partial y_l} = p_l - \lambda \frac{\partial F(y)}{\partial y_l} = 0 \text{ for } l = 1, .., L, \text{ or } p = \lambda \nabla F(y^*) \tag{1}$$

$$\lambda \geq 0$$

$$\lambda F(y) = 0$$

$$F(y) \leq 0$$

[2]Suppose $F(\cdot)$ is differentiable.

Claim. $F(y) = 0$.

Proof. If $F(y) < 0$, then $\exists y' \gg y$ and $F(y') < 0$. Since y' is feasible and $p \cdot y' > p \cdot y$, it contradicts with the fact that y maximizes profit.

- Equation (1) implies $\frac{p_l}{p_k} = \frac{\partial F(y^*)/\partial y_l}{\partial F(y^*)/\partial y_k} = MRT_{lk}(y^*)$.

- In the case of single-output production, profit $= pf(z) - w \cdot z$ where $p > 0$ is a scalar, and $w = (w_1, ..., w_{L-1}) \gg 0$ is a vector of input prices.

- The profit maximization problem for single-output production is

$$\max_{z \geq 0, q \geq 0} pq - w \cdot z$$
$$\text{s.t. } q \leq f(z)$$

Clearly, the constraint must hold in equality, since otherwise one can increase the production scale q without violating the constraint and earn a higher profit. Therefore, the above profit maximization problem could equivalently be written as

$$\max_{z \geq 0} pf(z) - w \cdot z$$

Remark. Here $z \geq 0$ is required but not in the previous configuration ($y \in \mathbb{R}^L$).

- Lagrange Function:

$$\mathcal{L} = pf(z) - w \cdot z$$

- Kuhn-Tucker Conditions:

$$p\frac{\partial f(z^*)}{\partial z_l} - w_l \leq 0, \text{ with equality if } z_l^* > 0, \text{ for } l = 1, ..., L-1. \tag{2}$$
$$z^* \geq 0$$

Equation (2) is equivalent to $p\nabla f(z^*) \leq w$ and $[p\nabla f(z^*) - w] \cdot z^* = 0$.

- Suppose $(z_l^*, z_k^*) \gg 0$. Then,

$$p\frac{\partial f(z^*)}{\partial z_l} = w_l \text{ and } p\frac{\partial f(z^*)}{\partial z_k} = w_k$$
$$\implies \frac{w_l}{w_k} = \frac{\partial f(z^*)/\partial z_l}{\partial f(z^*)/\partial z_k} = MRTS_{lk}(z^*). \tag{3}$$

- Condition (3) can also be rewritten as

$$\frac{1}{w_l}\frac{\partial f(z^*)}{\partial z_l} = \frac{1}{w_k}\frac{\partial f(z^*)}{\partial z_k} = \text{ marginal product of \$1.}$$

In other words, when profit is maximized, the marginal product of \$ 1 of production cost spent on each input should be equal. Or else the same production cost should be spent on the input generating a higher marginal product per dollar. It is possible input j generates lower marginal product per dollar than the rest because this input is particularly ineffective. In that case, z_j^* must be zero. Note that the Kuhn-Tucker conditions accommodate this.

- If the production set Y is convex, then the F.O.C in (1) and (2) are not only necessary but also sufficient.

Exercise 5.C.9

Derive the profit function $\pi(p)$ and supply function (or correspondence) $y(p)$ for the single-output technologies whose production functions $f(z)$ are given by

(a) $f(z) = \sqrt{z_1 + z_2}$.

(b) $f(z) = \sqrt{\min\{z_1, z_2\}}$.

(c) $f(z) = (z_1^\rho + z_2^\rho)^{1/\rho}$, for $\rho \leq 1$.

Mathematical Appendix: Separating Hyperplane Theorem Now we need to visit the Mathematical Appendix to retrieve a result that we'll use to prove our next proposition for this chapter.

Theorem M.G.2 (Separating Hyperplane Theorem (Part I))**.** *Suppose that $\mathcal{B} \subset \mathbb{R}^N$ is convex and closed, and that $y \notin \mathcal{B}$. Then there is a $p \in \mathbb{R}^N$ with $p \neq 0$, and a value $c \in \mathbb{R}$ such that $p \cdot y > c$ and $p \cdot x < c$ for every $x \in \mathcal{B}$.*

Proof. For any $z \in \mathbb{R}^N$ and $z \neq y$, define $p = y - z$. First, $p \cdot y > p \cdot z$ because $p \cdot (y - z) = \|y - z\|^2 > 0$. Let $c = p \cdot \left(\frac{y+z}{2}\right)$ so that $p \cdot y > c > p \cdot z$.

Suppose $z = \arg\min_{x \in \mathcal{B}} \|y - x\|^2$. (see Figure 10) Consider an arbitrary $x \in \mathcal{B}$.

$$\begin{aligned}
\|z-y\|^2 \leq \|(1-\lambda)\,z + \lambda x - y\|^2 &= \|(1-\lambda)\,(z-y) + \lambda\,(x-y)\|^2 \\
&= (1-\lambda)^2\,\|z-y\|^2 + \lambda^2\,\|x-y\|^2 + 2\,(1-\lambda)\,\lambda\,(z-y)\cdot(x-y) \\
\implies\ 0 &\leq \lambda\,(\lambda-2)\,\|z-y\|^2 + \lambda^2\,\|x-y\|^2 + 2\,(1-\lambda)\,\lambda\,(z-y)\cdot(x-y) \\
\implies\ 0 &\leq (\lambda-2)\,\|z-y\|^2 + \lambda\,\|x-y\|^2 + 2\,(1-\lambda)\,(z-y)\cdot(x-y)
\end{aligned}$$

Taking limit, letting λ go to zero,

$$\begin{aligned}
0 &\leq -2\,(z-y)\cdot(z-y) + 2\,(z-y)\cdot(x-y) \\
0 &\leq 2\,(z-y)\cdot(x-z) = -2p\cdot(x-z) \\
p\cdot z &\geq p\cdot x.
\end{aligned}$$

Therefore, $p \cdot y > c > p \cdot z \geq p \cdot x$ for all $x \in \mathcal{B}$. □

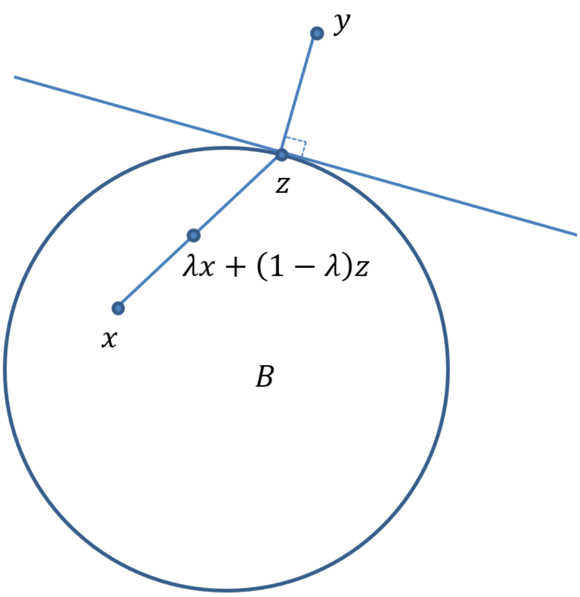

Figure 10: Seperating Hyperplane

Proposition 5.C.1. *Suppose $\pi(\cdot)$ is the profit function of the production set Y and that $y(\cdot)$ is the associated supply correspondence. Assume also that Y is closed and satisfies the free disposal property. Then,*

(i) $\pi(\cdot)$ is homogeneous of degree one.

(ii) $\pi(\cdot)$ is convex.

(iii) *If Y is convex, then* $Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0\}$.

(iv) $y(\cdot)$ *is homogeneous of degree zero.*

(v) *If Y is convex, then $y(p)$ is a convex set for all p. Moreover, if Y is strictly convex, then $y(p)$ is single-valued (if nonempty).*

(vi) *(Hotelling's lemma) If $y(\bar{p})$ consists of a single point, then $\pi(\cdot)$ is differentiable at $\bar{p}$ and $\nabla\pi(\bar{p}) = y(\bar{p})$.*

(vii) *If $y(\cdot)$ is a function differentiable at $\bar{p}$, then $Dy(\bar{p}) = D^2\pi(\bar{p})$ is a symmetric and positive semidefinite matrix with $Dy(\bar{p})\bar{p} = 0$.*

Proof.

(i) & (iv) The solution to

$$\max_{y \in \mathbb{R}^L} \alpha p \cdot y$$
$$\text{s.t. } y \in Y$$

and

$$\max_{y \in \mathbb{R}^L} p \cdot y$$
$$\text{s.t. } y \in Y$$

are identical. Therefore, $y(\alpha p) = y(p)$. This proves (iv).

Next, $\pi(\alpha p) = (\alpha p) \cdot y(\alpha p) = \alpha p \cdot y(p) = \alpha \pi(p)$. This proves (i).

(ii) $\pi(p) = p \cdot y(p) \geq p \cdot \tilde{y}$ for any $\tilde{y} \in Y$. $\pi(p') = p' \cdot y(p') \geq p' \cdot \tilde{y}$ for any $\tilde{y} \in Y$.

$$\begin{aligned}
\pi(\alpha p + (1-\alpha)p') &= [\alpha p + (1-\alpha)p'] \cdot y(\alpha p + (1-\alpha)p') \\
&= \alpha p \cdot y(\alpha p + (1-\alpha)p') + (1-\alpha)p' \cdot y(\alpha p + (1-\alpha)p') \\
&\leq \alpha p \cdot y(p) + (1-\alpha)p' \cdot y(p') \\
&= \alpha\pi(p) + (1-\alpha)\pi(p')
\end{aligned}$$

Therefore, $\pi(\cdot)$ is convex.

Intuition: Consider the scenario in which the price is uncertain; with probability α it is p and with probability $(1-\alpha)$ it is p'. If the firm chooses output under this uncertainty, its output is $y(\alpha p + (1-\alpha)p')$ and profit is $\pi(\alpha p + (1-\alpha)p')$. The result is same as when the price is fixed at $\alpha p + (1-\alpha)p'$. It is intuitive that the firm's profit would be higher if it gets to know the realization of the price before choosing y. In that case, its expected profit is $\alpha\pi(p) + (1-\alpha)\pi(p')$.

(iii) If Y is convex and closed, then by Theorem M.G.2 Separating Hyperplane Theorem (Part I), $\forall x \notin Y$, there exists $p \neq 0$ s.t. $p \cdot x > p \cdot y$ for all $y \in Y$.

Note that $\pi(p) = p \cdot y^* \geq p \cdot y$ for some $y^* \in Y$ and for all $y \in Y$. Therefore, $p \cdot x > \pi(p) \geq p \cdot y$.

Now we establish that $p \geq 0$. Suppose $p_l < 0$ for some l. Then by free disposal, $y - \theta e_l \in Y$ for any $\theta > 0$. This implies that an arbitrarily large profit can be achieved by choosing θ sufficiently large. This contradicts $p \cdot x > p \cdot (y - \theta e_l)$.

Next, we argue that it is without loss of generality to focus on $p \gg 0$. Suppose $p_l = 0$ for some l. Then there exists $\alpha > 0$ sufficiently small such that $(p + \alpha e_l) \cdot x > \pi(p + \alpha e_l) \geq (p + \alpha e_l) \cdot y$ for all $y \in Y$.[3] Now the new price vector $p' = p + \alpha e_l$ satisfies $p'_l = p_l + \alpha e_l > 0$. We could apply the same procedure for all $p_l = 0$. The resulting price vector satisfies $\tilde{p} \gg 0$.

We have proved that each $x \notin Y$ (no matter how close to Y) is excluded from the half space $p \cdot y \leq \pi(p)$ for some $p \gg 0$. Then the intersection of such half spaces for all $p \gg 0$ *excludes all* $x \notin Y$. However, all such half spaces include Y so *their intersection also includes* Y.

Therefore, $Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0\}$.

(v) Suppose $y, y' \in y(p) \subset Y$. Then,

$$p \cdot y = p \cdot y' = \pi(p) \text{ and } F(y) \leq 0, F(y') \leq 0$$
$$\Longrightarrow p \cdot (\alpha y + (1-\alpha)y') = \alpha p \cdot y + (1-\alpha)p \cdot y' = \pi(p), \ \forall \alpha \in [0,1]$$

[3] The first inequality follows from continuity of $\pi(\cdot)$ and the second inequality follows from the definition of $\pi(\cdot)$.

Convexity of Y implies $\alpha y + (1-\alpha)y' \in Y$. Therefore, $y(p)$ is convex.

Suppose Y is strictly convex. Consider $y, y' \in y(p)$ and $y \neq y'$. Then, $F(\alpha y + (1-\alpha)y') < 0$ for $\alpha \in (0,1)$. Then, $\exists\, y'' \gg \alpha y + (1-\alpha)y'$ s.t. $F(y'') \leq 0$, and $p \cdot y'' > p \cdot (\alpha y + (1-\alpha)y') = \alpha p \cdot y + (1-\alpha) p \cdot y' = \pi(p)$.

This contradicts the definition of $\pi(p)$.

(vi) Proof of differentiability of $\pi(p)$ is skipped.

The profit maximization problem could be written as

$$\max_{y \in \mathbb{R}^{\mathbb{L}}} p \cdot y$$
$$\text{s.t. } F(y) = 0$$

The maximized profit is $\pi(p) = p \cdot y^*$, where y^* is the solution to the problem.

Lagrange Function:

$$\mathcal{L}(y, \lambda) = p \cdot y - \lambda F(y)\,.$$

By Envelope Theorem,

$$\frac{\partial(\pi(\bar{p}))}{\partial p_l} = \frac{\partial \mathcal{L}(y^*, \lambda^*, \bar{p})}{\partial p_l} = y_l^* = y_l(\bar{p}).$$

In matrix notation, $\nabla \pi(\bar{p}) = y(\bar{p})$.

(vii)
- $Dy(\bar{p}) = D^2\pi(\bar{p})$ follows (vi) directly.
- Symmetry of $D^2\pi(\bar{p})$ is also standard for $\pi(p)$ being C^2 [Schwarz' theorem].
- Positive semidefiniteness follows from $\pi(p)$ being convex (ii). Taylor expansion:

  $$\pi(p + \alpha z) = \pi(p) + \nabla\pi(p) \cdot (az) + \frac{1}{2}(\alpha z) D^2\pi(p + \beta z)(\alpha z) \text{ for some } \beta \in [0, \alpha].$$
  $$\implies \frac{\alpha^2}{2} z^T D^2\pi(p + \beta z) z = \pi(p + \alpha z) - \pi(p) - \nabla\pi(p) \cdot (az) \geq 0 \;(\because \pi \text{ is convex.})$$

  This holds true for α, β arbitrarily small.

  $$\implies z^T D^2\pi(p) z \geq 0.$$

- By (iv), $y(\alpha p) = y(p)$ (H.D.$\emptyset$).

  Differentiating both sides of the equation by α gives: $Dy(\alpha p)p = 0$.

  Setting $\alpha = 1$, we have $Dy(p)p = 0$. □

Remark. $\nexists$ budget constraint, so no "income" effect associated with price change.

Law of Supply

Claim. $(p - p') \cdot (y - y') \geq 0$ [That is, $dp \cdot dy = dp^T D y dp \geq 0$]

Proof. $(p - p') \cdot (y - y') = (p \cdot y - p \cdot y') + (p' \cdot y' - p' \cdot y) \geq 0.$ □

Cost Minimization

- Cost minimization is necessary (but not sufficient) for profit maximization.
- We focus on single-output production.
- Cost Minimization Problem (CMP):

$$\begin{aligned} &\min_{z \geq 0} w \cdot z \\ &\text{s.t. } f(z) \geq q \end{aligned} \quad \equiv \quad \begin{aligned} &\max_{z \geq 0} \; - w \cdot z \\ &\text{s.t. } \; - f(z) \leq -q \end{aligned}$$

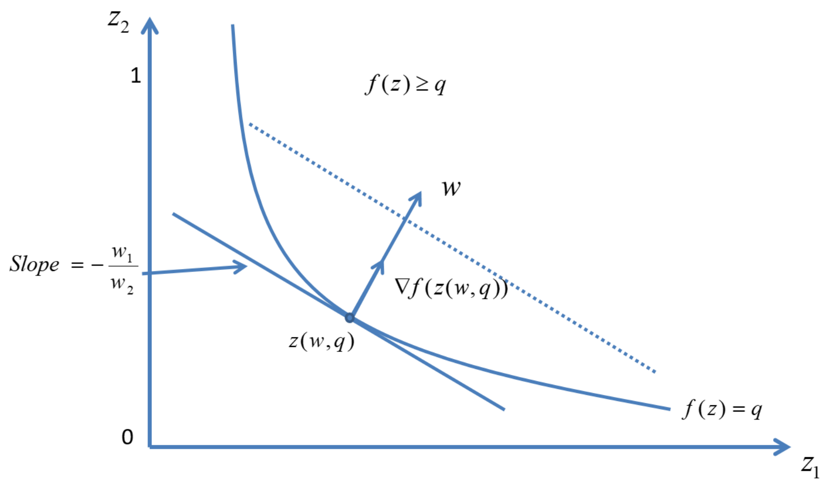

Figure 11: CMP for Single-output Production

- Let $z(w, q)$ denote the solution of CMP, $c(w, q)$ denote the minimized cost, or the cost function. $z(w, q)$ is known as the *conditional factor demand function* or correspondence.
- Lagrange Function:

$$\mathcal{L} = (-w \cdot z) - \lambda(-f(z) + q)$$

- Kuhn-Tucker Conditions:

$$-w_l + \lambda \frac{\partial f(z^*)}{\partial z_l} \leq 0 \Longleftrightarrow w_l \geq \lambda \frac{\partial f(z^*)}{\partial z_l}, \text{with equality if } z_l^* > 0 \tag{4}$$

$$\lambda \geq 0$$

$$\lambda(-f(z) + q) = 0$$

$$-f(z) \leq -q$$

$$z \geq 0$$

- Equation (4) is equivalent to $w \geq \lambda \nabla f(z^*)$ and $[w - \lambda \nabla f(z^*)] \cdot z^* = 0$.

- For any l, k with $(z_l, z_k) \gg 0$, we have

$$\frac{w_l}{w_k} = \frac{\partial f(z^*)/\partial z_l}{\partial f(z^*)/\partial z_k} = MRTS_{lk}$$

- λ measures $\partial c(w, q)/\partial q$, or the marginal cost of production.

- As with Profit Maximization Problem, if the production set Y is convex, then F.O.C. (Equation (4)) is not only necessary but also sufficient for z^* to be an optimum in Cost Minimization Problem.

Proposition 5.C.2. *Suppose that $c(w, q)$ is the cost function of a single-output technology Y with production function $f(\cdot)$ and that $z(w, q)$ is the associated conditional factor demand correspondence. Assume also that Y is closed and satisfies the free disposal property. Then,*

(i) *$c(\cdot)$ is homogeneous of degree one in w and nondecreasing in q.*

(ii) *$c(\cdot)$ is a concave function of w.*

(iii) *If the sets $\{z \geq 0 : f(z) \geq q\}$ are convex for every q, then $Y = \{(-z, q) : w \cdot z \geq c(w, q) \text{ for all } w \gg 0\}$.*

(iv) *$z(\cdot)$ is homogeneous of degree zero in w.*

(v) *If the set $\{z \geq 0 : f(z) \geq q\}$ is convex, then $z(w, q)$ is a convex set. Moreover, if $\{z \geq 0 : f(z) \geq q\}$ is a strictly convex set, then $z(w, q)$ is single-valued.*

(vi) *(Shepard's lemma) If $z(\bar{w}, q)$ consists of a single point, then $c(\cdot)$ is differentiable with respect to w at $\bar{w}$ and $\nabla_w c(\bar{w}, q) = z(\bar{w}, q)$.*

(vii) *If $z(\cdot)$ is differentiable at $\bar{w}$, then $D_w z(\bar{w}, q) = D_w^2 c(\bar{w}, q)$ is symmetric and negative semidefinite matrix with $D_w z(\bar{w}, q)\bar{w} = 0$.*

(viii) *If $f(\cdot)$ is homogeneous of degree one (i.e., exhibits constant returns to scales), then $c(\cdot)$ and $z(\cdot)$ are homogeneous of degree one in q.*

(ix) *If $f(\cdot)$ is concave, then $c(\cdot)$ is a convex function of q (in particular, marginal costs are nondecreasing in q).*

Remark. Note that cost minimization is very similar to expenditure minimization.

Proof.

(i) & (iv) The cost minimization problem

$$\begin{array}{c} \min\limits_{z \geq 0} \alpha w \cdot z \\ \text{s.t. } f(z) \geq q \end{array} \equiv \begin{array}{c} \min\limits_{z \geq 0} w \cdot z \\ \text{s.t. } f(z) \geq q \end{array}$$

Therefore, $z(\alpha w, q) = z(w, q)$, or $z(w, q)$ is H.D.$\emptyset$ in w, which is (iv). For (i),

$$c(\alpha w, q) = aw \cdot z(\alpha w, q) = aw \cdot z(w, q) = \alpha c(w, q).$$

Therefore, $c(w, q)$ is H.D.1 in w.

Next, we prove that $c(\cdot)$ is nondecreasing in q (the second part of (i)).

Suppose $q'' > q'$. Let the input price be w, and z'' and z' be the optimal input bundles for output levels q'' and q' respectively.

Since $f(z'') \geq q'' > q'$, definition of $c(\cdot)$ implies $c(w, q') \leq w \cdot z'' = c(w, q'')$.

(ii) To prove concavity in w, we need to show

$$c(\alpha w + (1 - \alpha)w', q) \geq \alpha c(w, q) + (1 - \alpha)c(w', q) \ \forall \alpha \in [0, 1].$$

To see this,

$$c(\alpha w+(1-\alpha)w',q)=[\alpha w+(1-\alpha)w']\cdot z(\alpha w+(1-\alpha)w',q)$$
$$=\alpha w\cdot z(\alpha w+(1-\alpha)w',q)+(1-\alpha)w'\cdot z(\alpha w+(1-\alpha)w',q)$$
$$\geq \alpha w\cdot z(w,q)+(1-\alpha)w'\cdot z(w',q)$$
$$=\alpha c(w,q)+(1-\alpha)c(w',q)$$
$$\Longrightarrow c(\cdot) \text{ is concave in } w.$$

(iii) Given that $\widetilde{Y}_q=\{z\geq 0 : f(z)\geq q\}$ is convex and closed, by Theorem M.G.2 Separating Hyperplane Theorem (Part I), $\forall x\notin \widetilde{Y}_q$, there exists $w\neq 0$ s.t. $w\cdot x< w\cdot z$, $\forall z\in\widetilde{Y}_q$.

Note that $c\left(w,q\right)=w\cdot z^*\leq w\cdot z$ for some $z^*\in\widetilde{Y}_q$ and for all $z\in\widetilde{Y}_q$. So $w\cdot x<c\left(w,q\right)\leq w\cdot z$.

Now, we show that $w\geq 0$. Suppose $w_l<0$ for some l. In this case, by free disposal, $f\left(z+\theta e_l\right)\geq q$, so $z+\theta e_l\in\widetilde{Y}_q$ for $\theta>0$. For all x, there exists $\theta>0$ sufficiently large such that $w\cdot\left(z+\theta e_l\right)<w\cdot x$. This contradicts $w\cdot x<w\cdot z,\forall z\in\widetilde{Y}_q$.

Next, we argue that it is without loss of generality to restrict attention to $w\gg 0$. Suppose $w_l=0$ for some l. In this case, there exists $\alpha>0$ sufficiently small such that $\left(w+\alpha e_l\right)\cdot x<\left(w+\alpha e_l\right)\cdot z$ for all $z\in\widetilde{Y}_q$.

Since every $x\notin Y_q$ is excluded by some half space $w\cdot z\geq c(w,q)$ for some $w\gg 0$, the intersection of all such half spaces for all $w\gg 0$ excludes all $x\notin Y_q$. On the other hand, the intersection still covers $\widetilde{Y}_q$. Therefore, $\widetilde{Y}_q=\{z\in\mathbb{R}^{L-1} : w\cdot z\geq c(w,q)$ for all $w\gg 0\}$. Since $y_L=q$ and $(y_1,...,y_{L-1})=-z$, $Y=\{(-z,q) : w\cdot z\geq c(w,q)$ for all $w\gg 0\}$.

(v) Suppose $z_1,z_2\in z(w,q)$. Then,

$$w\cdot z_1=w\cdot z_2=c(w,q)\leq w\cdot z,\forall z\in Y_q\equiv\{z\geq 0 : f(z)\geq q\}$$
$$\Longrightarrow w\cdot(\alpha z_1+(1-\alpha)z_2)=\alpha w\cdot z_1+(1-\alpha)w\cdot z_2=c(w,q).$$

Since Y_q is convex, $(\alpha z_1+(1-\alpha)z_2)\in Y_q$. Therefore, $\alpha z_1+(1-\alpha)z_2\in z(w,q)$.

Suppose $z_1, z_2 \in z(w,q)$ and $z_1 \neq z_2$. If Y_q is strictly convex, then $f(\alpha z_1 + (1-\alpha)z_2) > q$ and there exists $\theta \in (0,1)$ such that $f(\theta(\alpha z_1 + (1-\alpha)z_2)) \geq q$ and $w \cdot \theta(\alpha z_1 + (1-\alpha)z_2) < c(w,q)$. This contradicts the definition of $c(w,q)$.

(vi) Proof of differentiability of $c(\cdot)$ is skipped.

The Lagrange Function:

$$\mathcal{L}(z, \lambda, \bar{w}) = -\bar{w} \cdot z - \lambda \left(-f(z) + q\right).$$

By Envelope Theorem,

$$\frac{\partial c(\bar{w},q)}{\partial w_l} = \frac{\partial \mathcal{L}(z^*, \lambda^*, \bar{w})}{\partial w_l} = z_l^* = z_l(\bar{w},q)$$

In matrix notation, $\nabla_w c(\bar{w},q) = z(\bar{w},q)$.

(vii)
- $D_w z(\bar{w},q) = D_w^2 c(\bar{w},q)$ follows differentiability of $z(\cdot)$ and (vi) immediately.
- Symmetry of $D_w^2 c(\bar{w},q)$ is standard for $c(\bar{w},q)$ being C^2 [Schwarz' theorem].
- Positive semidefiniteness follows from $c(w,q)$ being concave in w (ii).

  Taylor expansion:

  $$c(w+\alpha v, q) = c(w,q) + D_w c(w,q)\alpha v + \frac{1}{2}(\alpha v) D_w^2 c(w+\beta v, q)(\alpha v) \text{ for } \beta \in [0,\alpha]$$
  $$\implies \frac{\alpha^2}{2} v^T D_w^2 c(w+\beta v, q) v = c(w+\alpha v, q) - c(w,q) - D_w c(w,q)\alpha v \leq 0$$
  $$\because c \text{ is concave in } w.$$

  This holds true for α, β arbitrarily small. $\implies v^T D_w^2 c(w,q) v \leq 0$.
- by (iv), $z(\alpha w, q) = z(w,q)$ (H.D.$\emptyset$ in w).

  Differentiating both sides of the equation by α gives: $D_w z(\alpha w, q) w = 0$.

  Setting $\alpha = 1$, we have $D_w z(w,q) w = 0$.

(viii) Note that $z(w, \lambda q)$ solves CMP$_1$:

$$\min_z w \cdot z$$
$$\text{s.t. } f(z) \geq \lambda q.$$

Since $f(z)$ is H.D.1,

$$f(z) \geq \lambda q \iff \frac{f(z)}{\lambda} \geq q \iff f\left(\frac{z}{\lambda}\right) \geq q.$$

CMP_1 can be stated as CMP_2

$$\min_{\tilde{z}} w \cdot (\tilde{z})$$
$$\text{s.t. } f(\tilde{z}) \geq q,$$

where $\tilde{z} = z/\lambda$. Let $\tilde{z}^*$ be the solution to CMP_2. Then

$$\frac{z(w, \lambda q)}{\lambda} = \tilde{z}^* = z(w, q)$$
$$\implies z(w, \lambda q) = \lambda z(w, q).$$

That is, $z(\cdot)$ is H.D.1 in q.

An alternative proof of $z(w, \alpha q) = \alpha z(w, q)$

- It is equivalent to show that $\alpha z(w, q)$ is the solution to the cost minimization problem with parameters $(w, \alpha q)$.
  a) Since $f(\cdot)$ is HD1, $f(\alpha z(w, q)) = \alpha f(z(w, q)) \geq \alpha q$. Thus, $\alpha z(w, q)$ satisfies the constraint of the $(w, \alpha q)$ problem.
  b) For any z such that $f(z) \geq \alpha q$, since $f(\cdot)$ is HD1, we have $f(z) \geq \alpha q \implies \alpha^{-1} f(z) \geq q \implies f(\alpha^{-1} z) \geq q$. That is, $\alpha^{-1} z$ satisfies the constraint of the (w, q) problem. Thus, the cost of $\alpha^{-1} z$ must be weakly higher than the minimum cost which is obtained at $z(w, q)$, that is, $w \cdot (\alpha^{-1} z) \geq w \cdot z(w, q)$. We further have $w \cdot z \geq w \cdot (\alpha z(w, q))$.
- Since a) $\alpha z(w, q)$ satisfies the constraint of the $(w, \alpha q)$ problem and b) $w \cdot (\alpha z(w, q)) \leq w \cdot z$ for any z such that $f(z) \geq \alpha q$, $\alpha z(w, q)$ is the solution to the cost minimization problem with parameters $(w, \alpha q)$.
- Thus, $\alpha z(w, q) = z(w, \alpha q)$.

For $c(\cdot)$,

$$c(w, \alpha q) = w \cdot z(w, \alpha q) = \alpha w \cdot z(w, q) = \alpha c(w, q).$$

That is, $c(\cdot)$ is also H.D.1 in q.

(ix) Let $z_1 = z(w, q_1)$ and $z_2 = z(w, q_2)$ be the solutions to cost minimization problems with parameters (w, q_1) and (w, q_2) respectively. Let $c(w, q_1) = w \cdot z_1$ and $c(w, q_2) = w \cdot z_2$ be the minimized cost.

- First, $\alpha z_1 + (1-\alpha)z_2$ satisfies the constraint of the cost minimization problem with parameters $(w, \alpha q_1 + (1-\alpha)q_2)$:

$$f(\alpha z_1 + (1-\alpha)z_2) \underbrace{\geq}_{\text{concavity}} \alpha f(z_1) + (1-\alpha)f(z_2) \underbrace{\geq}_{f(z_1)\geq q_1,\, f(z_2)\geq q_2} \alpha q_1 + (1-\alpha)q_2.$$

- Thus, the cost of $\alpha z_1 + (1-\alpha)z_2$ must be higher than the minimum cost $c(w, \alpha q_1 + (1-\alpha)q_2)$, that is, $c(w, \alpha q_1 + (1-\alpha)q_2) \leq w \cdot (\alpha z_1 + (1-\alpha)z_2)$.

- Therefore,

$$\begin{aligned} c(w, \alpha q_1 + (1-\alpha)q_2) &\leq w \cdot (\alpha z_1 + (1-\alpha)z_2) \\ &= \alpha w \cdot z_1 + (1-\alpha)w \cdot z_2 \\ &\underbrace{=}_{\text{definition of } z_1 \,\&\, z_2} \alpha c(w, q_1) + (1-\alpha)c(w, q_2) \end{aligned}$$

That is, $c(\cdot)$ is convex in q. $\square$

From Cost Minimization to Profit Maximization We restate Profit Maximization Problem using the cost function:

$$\max_{q\geq 0}\ pq - c(w, q).$$

Kuhn-Tucker Conditions:

$$p - \frac{\partial c(w, q^*)}{\partial q} \leq 0 \text{ with equality if } q^* > 0 \tag{5}$$

$$q \geq 0.$$

Equation (5) indicates that at an interior optimum (i.e., if $q^* > 0$), *price equals marginal cost*. If $c(w, q)$ is convex in q, then the F.O.C (Equation (5)) is not only necessary but also sufficient for q^* to be the optimal production level.

Example 5.C.1. (Building on Example 5.B.2): Derive the cost and profit functions for the Cobb-Douglas production function $f(z_1, z_2) = z_1^\alpha z_2^\beta$.

Remark. Note that $f(\cdot)$ is constant returns to scale if $\alpha + \beta = 1$, increasing returns to scale if $\alpha + \beta > 1$, and decreasing returns to scale if $\alpha + \beta < 1$.

Solution. We first consider the cost minimization problem.

Cost minimization

$$\begin{aligned} \min_{z_1, z_2 \geq 0} \ & w_1 z_1 + w_2 z_2 \\ \text{s.t.} \ & z_1^\alpha z_2^\beta \geq q \end{aligned} \equiv \begin{aligned} \max_{z_1, z_2 \geq 0} \ & - w_1 z_1 - w_2 z_2 \\ \text{s.t.} \ & - z_1^\alpha z_2^\beta \leq -q \end{aligned}$$

Lagrange Function:

$$\mathcal{L} = -w_1 z_1 - w_2 z_2 - \lambda(-z_1^\alpha z_2^\beta + q)$$

Kuhn-Tucker Conditions:

$$\begin{aligned} & - w_1 + \lambda(\alpha z_1^{\alpha-1} z_2^\beta) \leq 0, \text{ with equality if } z_1 > 0 \\ & - w_2 + \lambda(\beta z_1^\alpha z_2^{\beta-1}) \leq 0, \text{ with equality if } z_2 > 0 \\ & \lambda \geq 0 \\ & \lambda(-z_1^\alpha z_2^\beta + q) = 0 \\ & - z_1^\alpha z_2^\beta \leq -q \\ & z_1, z_2 \geq 0 \end{aligned}$$

Note that for any $q > 0$, $z_1^* > 0$ & $z_2^* > 0$ must hold (if not, $z_1^{*\alpha} z_2^{*\beta} = 0 < q$). Therefore,

$$\frac{w_1}{w_2} = \frac{\alpha}{\beta} \frac{z_2}{z_1} \iff z_2 = z_1 \frac{w_1 \beta}{w_2 \alpha}. \tag{6}$$

Also, it must hold that $z_1^\alpha z_2^\beta = q$. If not, less of both inputs can be used and production cost can be lowered. Plugging (6) into this equality gives

$$\begin{aligned} z_1(w_1, w_2, q) &= q^{1/(\alpha+\beta)} (\frac{\alpha w_2}{\beta w_1})^{\beta/(\alpha+\beta)}; \\ z_2(w_1, w_2, q) &= q^{1/(\alpha+\beta)} (\frac{\beta w_1}{\alpha w_2})^{\alpha/(\alpha+\beta)}. \end{aligned}$$

It follows immediately that the (conditional) cost function is

$$\begin{aligned} c(w_1, w_2, q) &= w_1 z_1(w_1, w_2, q) + w_2 z_2(w_1, w_2, q) \\ &= q^{1/(\alpha+\beta)} [(\frac{\alpha}{\beta})^{\beta/(\alpha+\beta)} + (\frac{\alpha}{\beta})^{-\alpha/(\alpha+\beta)}] w_1^{\alpha/(\alpha+\beta} w_2^{\beta/(\alpha+\beta)} \\ &= q^{1/(\alpha+\beta)} \theta \phi(w_1, w_2), \end{aligned} \tag{7}$$

where $\theta = (\frac{\alpha}{\beta})^{\beta/(\alpha+\beta)} + (\frac{\alpha}{\beta})^{-\alpha/(\alpha+\beta)}$ and $\phi(w_1, w_2) = w_1^{\alpha/(\alpha+\beta} w_2^{\beta/(\alpha+\beta)}$.

Profit maximization Cost minimization is only a necessary condition for profit maximization. To maximize profit, the firm must choose the optimal quantity:

$$\max_{q \geq 0} pq - c(w,q)$$

Kuhn-Tucker Conditions:

$$p - \frac{\partial c(w,q^*)}{\partial q} \leq 0 \text{ with equality if } q^* > 0 \tag{8}$$

$$q \geq 0.$$

When (8) holds in equality, $p = MC$.

Plugging (7) into (8), we have

$$p \leq \theta\phi(w_1,w_2)(\frac{1}{\alpha+\beta})q^{1/(\alpha+\beta)-1}, \text{ with equality if } q > 0. \tag{9}$$

Case I When $\alpha + \beta < 1$, $f(\cdot)$ is concave and $c(\cdot)$ is convex in q, i.e., MC increases in q. $\implies$ F.O.C. is sufficient.

At $q = 0$, the R.H.S of (9) is zero and $p \leq 0$ must not hold. So (9) must hold in equality. Optimal q is unique:

$$q(w_1,w_2,q) = \left[(\alpha+\beta)\left(\frac{p}{\theta\phi(w_1,w_2)}\right)\right]^{\frac{\alpha+\beta}{1-\alpha-\beta}}$$

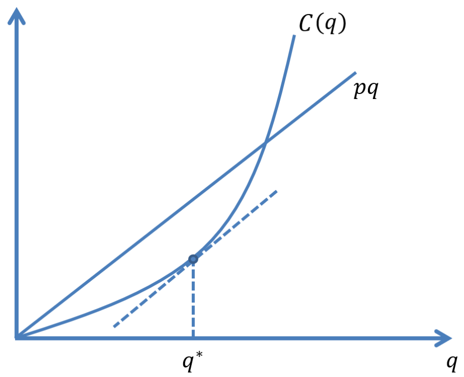

Figure 12: Case I

The factor demands can also be obtained through:

$$z_l(w_1,w_2,q) = z_l(w_1,w_2,q(w_1,w_2,p))$$

So can the profit function:

$$\pi(w_1,w_2,p) = pq(w_1,w_2,p) - w \cdot z(w_1,w_2,q(w_1,w_2,p))$$

Case II When $\alpha + \beta = 1$, (8) $\implies p \leq \theta\phi(w_1, w_2)$

(i) If $\theta\phi(w_1, w_2) > p$, then $q^* = 0$.

(ii) If $\theta\phi(w_1, w_2) < p$, then no solution: the higher q, the better.

(iii) If $\theta\phi(w_1, w_2) = p$, (knife-edge case): any nonnegative q is a solution.

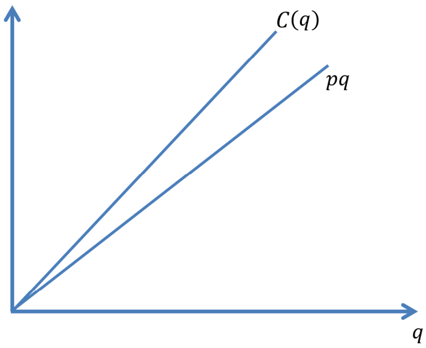

Figure 13: Case II (i)

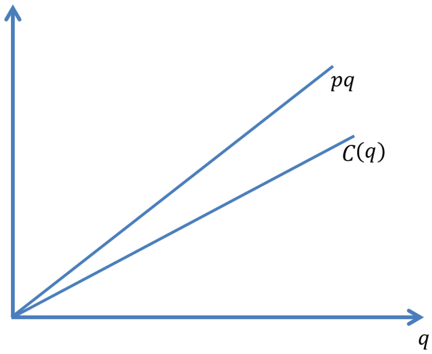

Figure 14: Case II (ii)

Case III When $\alpha + \beta > 1$, then F.O.C only identifies the local minimum.

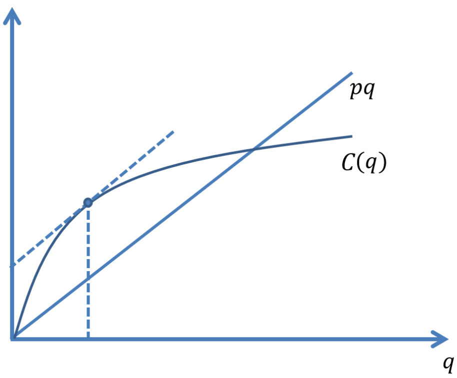

Figure 15: Case III

Exercise 5.C.10

Derive the cost function $c(w, q)$ and conditional factor demand functions (or correspondences) $z(w.q)$ for each of the following single-output constant return technologies with production functions given by

(a) $f(z) = z_1 + z_2$ (perfect substitutable inputs)

(b) $f(z) = \min\{z_1, z_2\}$ (leontief technology)

(c) $f(z) = (z_1^\rho + z_2^\rho)^{1/rho}, \rho \leq 1$ (constant elasticity of substitution technology)

Exercise 5.C.11

Show that $\partial z_l(w, q)/\partial q > 0$ if and only if marginal cost at q is increasing in w_l.

5.D. The Geometry of Cost and Supply on the Single-Output Case

Focusing on the single-output case, we analyze the relationships among: technology, cost function, and supply behavior.

We consider fixed factor prices $\bar{w} \gg 0$, and suppress the dependence on w, defining

$$C(q) = c(\bar{w}, q)$$

$$AC(q) = c(\bar{w}, q)/q$$

$$MC(q) = \partial c(\bar{w}, q)/\partial q$$

Convex Production Set Recall F.O.C for profit maximization: $p \leq C'(q)$, with equality if $q > 0$. If Y is convex, $c(\cdot)$ is convex and F.O.C is sufficient for profit maximization. An example of convex production set is given below:

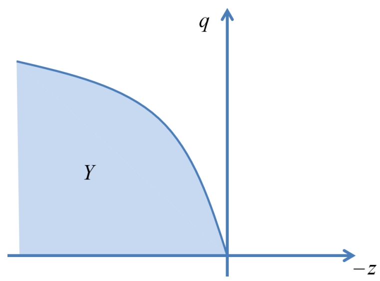

Figure 16: Production Set

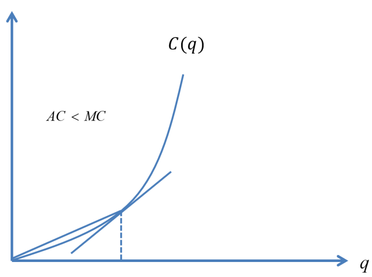

Figure 17: Cost Function

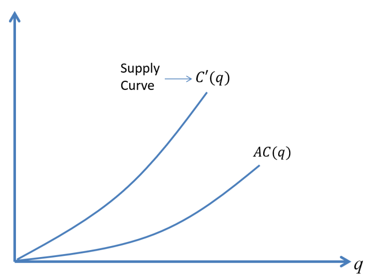

Figure 18: MC and AC

Nonconvex Production Set Y may not be convex. An example of nonconvex production set is given below:

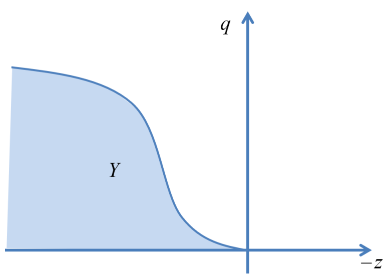

Figure 19: Production Set

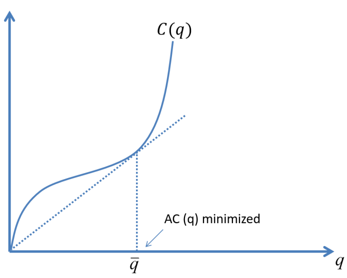

Figure 20: Cost Function

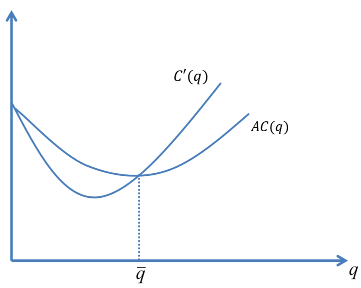

Figure 21: MC and AC

The relationship between Average Cost (AC) and Marginal Cost (MC):

$$AC(q) = c(q)/q$$
$$AC'(q) = \frac{qc'(q) - c(q)}{q^2}$$

The F.O.C. for minimization of AC is $\bar{q}c'(\bar{q}) - c(\bar{q}) = 0$ or $c'(\bar{q}) = \frac{c(\bar{q})}{\bar{q}}$, i.e., AC is minimized when $MC(\bar{q}) = AC(\bar{q})$.

Exercise 5.D.1

Show that $AC(\bar{q}) = C'(\bar{q})$ at any $\bar{q}$ satisfying $AC(\bar{q}) \leq AC(q)$ for all q. Does this result depend on the differentiability of $C(\cdot)$ everywhere?

Fixed cost (but not sunk) Fixed cost arises because some input(s) have to be used before any output can be produced. Since the cost is not sunk, it is still preventable so producing nothing and costing nothing is still an option.

For the firm to be willing to be active in production, the price has to at least cover the average cost of production. Otherwise, the firm will produce nothing.

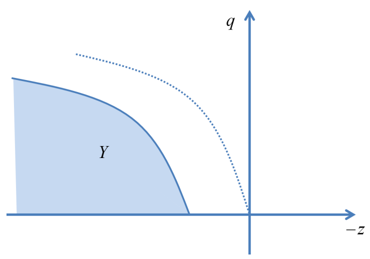

Figure 22: Production Set

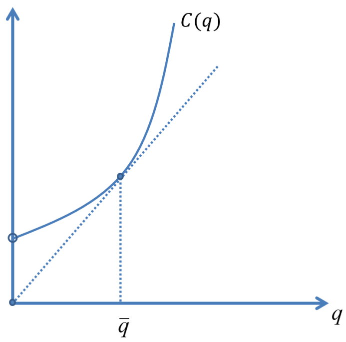

Figure 23: Cost Function

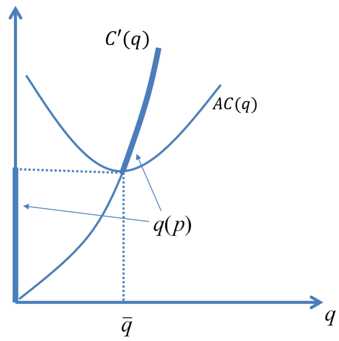

Figure 24: MC and AC

Sunk cost When cost is sunk, it is no longer preventable. So it is not an option to use no inputs and incur no cost. In deciding whether to be active in production or not, sunk cost should not be part of the consideration because by gone is by gone. Therefore, even if the price falls below the average cost, it may still be economically profitable (without accounting for the sunk cost) to be active in production.

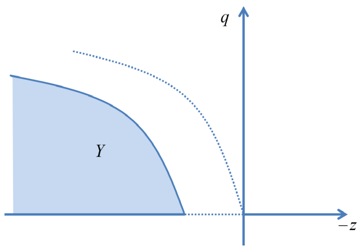

Figure 25: Production Set

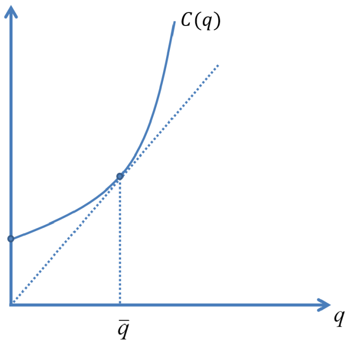

Figure 26: Cost Function

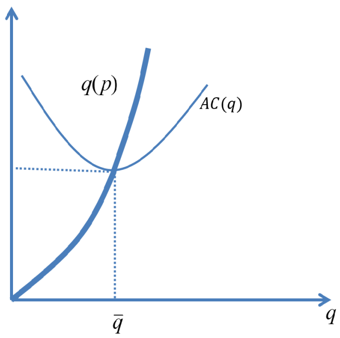

Figure 27: MC and AC

Exercise 5.D.2

Depict the supply locus for a case with partially sunk costs, that is, where $C(q) = K + C_v(q)$ if $q > 0$ and $0 < C(0) < K$.

Long-run and short-run cost functions In Figure 28, the cost function excluding any prior input commitments is depicted by $C(\cdot)$. We call it the *long-run cost function*. If one input, say z_2, is fixed at level $\bar{z}_2$ in the short-run, then the *short-run cost function* of the firm becomes $C(q \mid \bar{z}_2) = \bar{w}_1 z_1 + \bar{w}_2 \bar{z}_2$, where z_1, is chosen so that $f(z_1, \bar{z}_2) = q$.

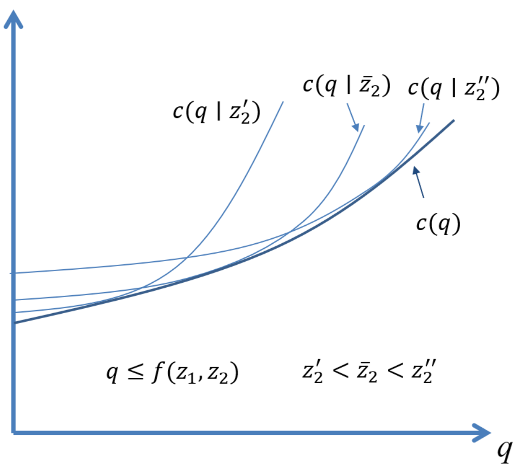

Figure 28: LR and SR Cost Functions

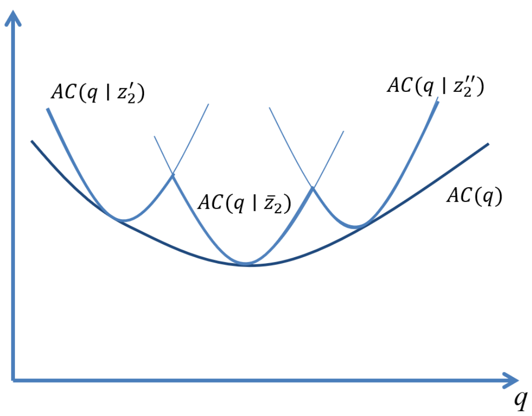

Figure 29: LR and SR AC

Exercise 5.D.3

Suppose that a firm can produce good L from $L-1$ factor inputs ($L > 2$). Factor prices are $w \in \mathbb{R}^{L-1}$ and the price of output is p. The firm's differentiable cost function is $c(w, q)$. Assume that this function is strictly convex in q. However, although $c(w, q)$ is the cost function when all factors can be freely adjusted, factor 1 cannot be adjusted in the short run.

Suppose that the firm is initially at a point where it is producing its long-run profit-maximizing output level of good L given prices w and p, $q(w, p)$ [i.e., the level that is optimal under the long-run cost conditions described by $c(w, q)$], and that all inputs are optimally adjusted [i.e., $z_l = z_l(w, q(w, p))$ for all $l = 1, ..., L-1$, where $z_l(\cdot, \cdot)$ is the long-run input demand function]. Show that the firm's profit-maximizing output response to a marginal increase in the price of good L is larger in the long run than in the short run. [*Hint*: Define a short-run cost function $c_s(w, q|z_1)$ that gives the minimized costs of producing output level q given that input 1 is fixed at level z_1.]

5.E. Aggregation

Question. Would the properties of individual supplies be preserved when they are aggregated to market supply?

Question. Would merger affect supply behavior?

- J production units/plants
- Y_j is nonempty, closed
- $\pi_j(p)$: profit function
- $y_j(p)$: supply correspondence
- Aggregate supply correspondence:

$$y(p) = \sum_{j=1}^{J} y_j(p) = \{y \in \mathbb{R}^L : y = \sum_j y_j \text{ for some } y_j \in y_j(p), j = 1, ..., J\}$$

- Suppose $y_j(p)$ is single-valued & differentiable.
  - From Proposition 5.C.1, $Dy_j(p)$ is symmetric & positive semidefinite.
  - Because these properties are preserved under addition, $Dy(p) = \sum_j Dy_j(p)$ is also symmetric and positive semidefinite.
  - Positive semidefiniteness implies law of supply in aggregate:

$$dp \cdot dy = dp^T Dy(p) dp \geq 0.$$

  - Alternatively, from $(p - p') \cdot [y_j(p) - y_j(p')] \geq 0$, adding over j, we have $(p - p') \cdot \left[\sum_j y_j(p) - \sum_j y_j(p')\right] \geq 0$.

- *Aggregate production set*:

$$Y = Y_1 + ... + Y_J = \{y \in \mathbb{R}^L : y = \sum_j y_j \text{ for some } y_j \in Y_j, j = 1, ..., J\}$$

  - Y is feasible to a single owner who maximizes total profit from J plants' production.
  - $\pi^*(p)$ and $y^*(p)$ are the profit function and the supply correspondence of Y.

Proposition 5.E.1. *For all $p \gg 0$, we have*

(i) $\pi^*(p) = \sum_j \pi_j(p)$

(ii) $y^*(p) = \sum_j y_j(p)$

Proof.

(i) a) First, we prove $\pi^*(p) \geq \sum_j \pi_j(p)$. (The owner of all J plants can at least replicate what the J individual owners do.)

- For any collection of production plans $y_j \in Y_j$ for $j = 1, ..., J$, we have $\sum_j y_j \in Y$.
- Since $\pi^*(p)$ is the profit function associated with Y, we have $\pi^*(p) \geq p \cdot \sum_j y_j = \sum_j p \cdot y_j = \sum_j \pi_j(p)$.

b) Next, we prove $\pi^*(p) \leq \sum_j \pi_j(p)$.

- By definition of Y, there exist $y_j \in Y_j$, $j = 1, ..., J$ such that $\sum_j y_j = y$.
- Therefore, for any $y \in Y$, $p \cdot y = p \cdot \sum_j y_j = \sum p \cdot y_j \leq \sum_j \pi_j(p)$.
- Therefore, $\pi^*(p) \leq \sum_j \pi_j(p)$.

Together, $\pi^*(p) \geq \sum_j \pi_j(p)$ and $\pi^*(p) \leq \sum_j \pi_j(p)$ imply $\pi^*(p) = \sum_j \pi_j(p)$.

(ii) We need to show $\sum_j y_j(p) \subseteq y^*(p)$ and $y^*(p) \subseteq \sum_j y_j(p)$.

a) First, we prove $\sum_j y_j(p) \subseteq y^*(p)$.

- Consider any set of individual production plans $y_j \in y_j(p)$ for $j = 1, ..., J$.
- Then $p \cdot \sum_j y_j = \sum_j p \cdot y_j = \sum_j \pi_j(p) = \pi^*(p)$ (the second equality follows from the definition of $\pi_j(p)$ and the last equality is by (i)).
- Therefore, $\sum_j y_j \in y^*(p)$, and thus $\sum_j y_j(p) \subseteq y^*(p)$.

b) Next, we prove $y^*(p) \subseteq \sum_j y_j(p)$.

- Take any $y \in y^*(p)$.
- By definition of Y, there exist $y_j \in Y_j$, $j = 1, ..., J$ such that $\sum_j y_j = y$.
- Thus, $\pi^*(p) = p \cdot y = p \cdot \sum_j y_j = \sum_j p \cdot y_j$.
- In addition, by (i), $\pi^*(p) = \sum_j \pi_j(p)$. Thus, $\sum_j \pi_j(p) = \sum_j p \cdot y_j$.
- By definition of $\pi_j(p)$, $p \cdot y_j \leq \pi_j(p)$. So it must be that $p \cdot y_j = \pi_j(p)$ for every $j = 1, ..., J$.
- Therefore, $y_j \in y_j(p)$ for all j.
- So $y = \sum_j y_j \in \sum_j y_j(p)$ and thus $y^*(p) \subseteq \sum_j y_j(p)$. □

Remark. This result that merger does affect supply behavior holds only because the firms are **price takers**. When these firms set prices to compete, the prices they set will have externality on each other's profit. After merger, the owner of all the plants typically will raise the prices to reduce the negative externality of low prices on other plants' profits.

5.F. Efficient Production (Narrow notion of efficiency)

> *Question.* When do we regard production as nonwasteful?

We take the prices as exogenously fixed and do not discuss whether the prices are too high or too low when we discuss the efficiency of a profit maximizing firm.

Definition 5.F.1. A production vector $y \in Y$ is efficient if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.

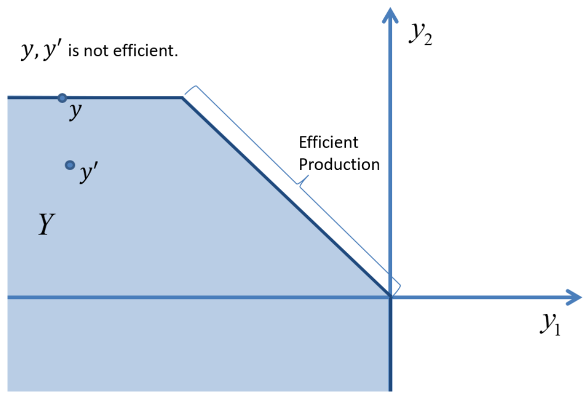

Figure 30: (In)Efficient Production

Proposition 5.F.1. *If $y \in Y$ is profit maximizing for some $p \gg 0$, then y is efficient.*

Proof. Suppose y is not efficient. Then $\exists y' \in Y$ s.t. $y' \geq y$ and $y' \neq y$. This implies $p \cdot y' > p \cdot y$ for all $p \gg 0$; so y is not profit maximizing. $\square$

Remark. Proposition 5.F.1 holds even when production set is non-convex. See Figure 31.

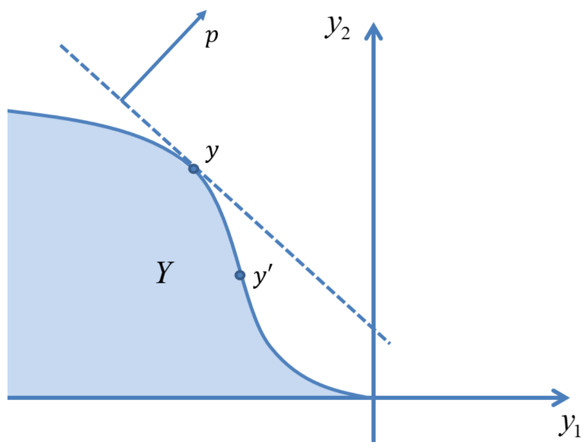

Figure 31: Non-convex production set

Exercise 5.F.1. Suppose $p_1 = 0$ & $p_2 > 0$. Then for all p_2, both y and y' maximize profit but y' is NOT efficient. This illustrates the importance of $p \gg 0$ in Proposition 5.F.1.

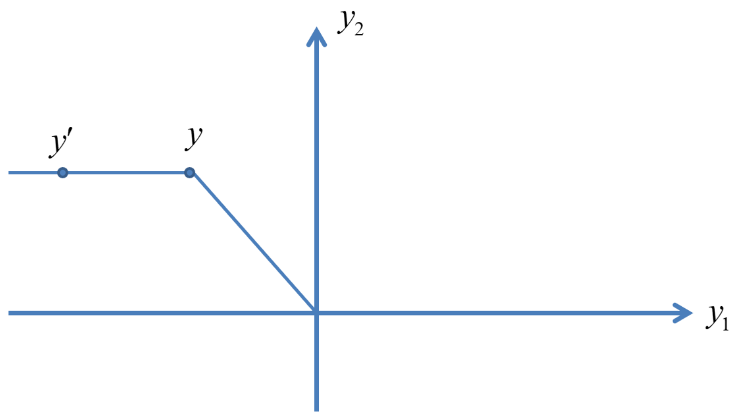

Figure 32: $p_1 = 0$

Now we need to visit the Mathematical Appendix to retrieve some results that we'll use to prove our next proposition for this chapter.

Theorem M.G.3 (Supporting Hyperplane Theorem)**.** *Suppose that $\mathcal{B} \subset \mathbb{R}^N$ is convex and that x is not an element of the interior of set $\mathcal{B}$ ($x \notin \operatorname{Int} \mathcal{B}$). Then there is $p \in \mathbb{R}^N$ with $p \neq 0$ such that $p \cdot x \geq p \cdot y$ for every $y \in \mathcal{B}$.*

Proof. Consider $x \notin \operatorname{Int} \mathcal{B}$. Then we can find a sequence $x^m \to x$ such that for all m, x^m is not an element of the *closure*[4] of the set $\mathcal{B}$ ($x^m \notin \operatorname{Cl} \mathcal{B}$). By Theorem M.G.2 Separating Hyperplane Theorem (Part I), for each m there is a $p^m \neq 0$ and a $c^m \in \mathbb{R}$ such that

$$p^m \cdot x^m > c^m \geq p^m \cdot y \tag{10}$$

for every $y \in \mathcal{B}$. Without loss of generality, suppose that $\|p^m\| = 1$ for every m. Thus, extracting a subsequence if necessary[5], we can assume that there is $p \neq 0$ and $c \in \mathbb{R}$ such that $p^m \to p$ and $c^m \to c$. Taking limits of (10), we have

$$p \cdot x \geq c \geq p \cdot y$$

for every $y \in \mathcal{B}$. □

[4] A *closure* of a set $\mathcal{A}$ is the union of the set $\mathcal{A}$ and its limit points.
[5] The existence of convergent subsequence is a result of the Bolzano–Weierstrass Theorem: each bounded sequence in $\mathbb{R}^N$ has a convergent subsequence.

Theorem M.G.2 (Separating Hyperplane Theorem (Part II)). *Suppose that the convex sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^N$ are disjoint (i.e., $\mathcal{A} \cap \mathcal{B} = \varnothing$). Then there is $p \in \mathbb{R}^N$ with $p \neq 0$, and a value $c \in \mathbb{R}$, such that $p \cdot x \geq c$ for every $x \in \mathcal{A}$ and $p \cdot y \leq c$ for every $y \in \mathcal{B}$. That is, there is a hyperplane that separates $\mathcal{A}$ and $\mathcal{B}$, leaving $\mathcal{A}$ and $\mathcal{B}$ on different sides of it.*

Proof. Consider arbitrary $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and let $z = x - y$. Let

$$\mathcal{D} = \left\{ z \in \mathbb{R}^N : z = x - y \text{ for some } x \in \mathcal{A} \text{ and some } y \in \mathcal{B} \right\}.$$

Now we show that $\mathcal{D}$ is convex. Suppose $z_1, z_2 \in \mathcal{D}$. Then

$$\alpha z_1 + (1-\alpha) z_2 = [\alpha x_1 + (1-\alpha) x_2] - [\alpha y_1 + (1-\alpha) y_2].$$

Since $\mathcal{A}$ and $\mathcal{B}$ are convex, $\alpha x_1 + (1-\alpha) x_2 \in \mathcal{A}$ and $\alpha y_1 + (1-\alpha) y_2 \in \mathcal{B}$.
So $\alpha z_1 + (1-\alpha) z_2 \in \mathcal{D}$. Therefore, $\mathcal{D}$ is convex. Since $\mathcal{A}$ and $\mathcal{B}$ are disjoint, $0 \notin \mathcal{D}$. Since $0 \notin \mathcal{D}$, we have $0 \notin \operatorname{Int} \mathcal{D}$. Then, we could apply Thereom M.G.3 Supporting Hyperplane Theorem: there is $p' \in \mathbb{R}^N$ with $p' \neq 0$ such that $p' \cdot 0 \geq p' \cdot z$ for all $z \in \mathcal{D}$. Let $p = -p'$, we have $0 \leq p \cdot (x - y)$ or $p \cdot y \leq p \cdot x$ for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$. To complete the proof, let

$$c = \frac{\inf_{x \in \mathcal{A}} p \cdot x + \sup_{y \in \mathcal{B}} p \cdot y}{2}. \quad \square$$

Proposition 5.F.2. *Suppose that Y is convex. Then every efficient production $y \in Y$ is a profit-maximizing production for some nonzero price vector $p \geq 0$.*

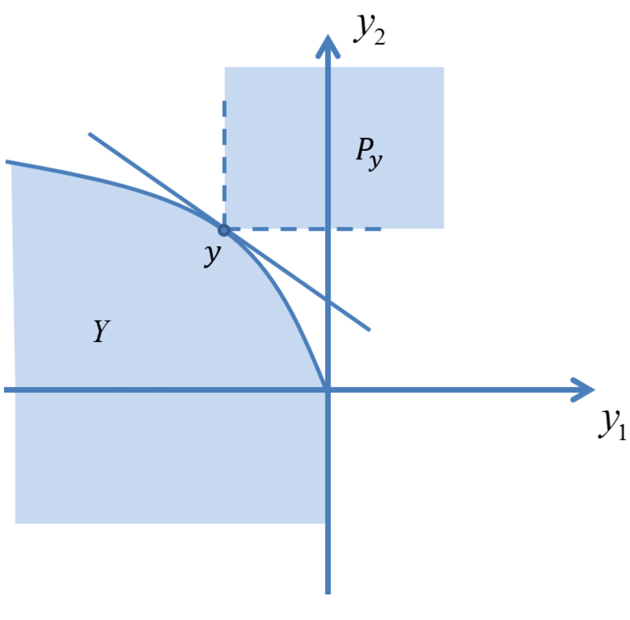

Figure 33: Proposition 5.F.2

Proof. Suppose $y \in Y$ is efficient and define $P_y = \{y' \in \mathbb{R}^L : y' \gg y\}$.

Step 1: we show that $\exists p \geq 0$ s.t. $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y$ and $y'' \in Y$.

- Since y is efficient, $\nexists x \in Y$, s.t. $x \geq y$ & $x \neq y$. Therefore, $Y \cap P_y = \varnothing$.
- Since P_y is convex and disjoint from Y, by Theorem M.G.2 Separating Hyperplane Theorem (Part II), $\exists p \neq 0$ s.t. $p \cdot y' \geq p \cdot y''$ for every $y' \in P_y$ and $y'' \in Y$.
- In particular, $p \cdot y' \geq p \cdot y$ for every $y' \gg y$. And for this to hold, it requires that $p \geq 0$. Suppose otherwise that $p_l < 0$ for some l. Then with y'_l sufficiently large, $p \cdot y' < p \cdot y$ necessarily holds, constituting a contradiction.

Step 2: we show that given $p \geq 0$ found in Step 1, we have $p \cdot y \geq p \cdot y''$ for every $y'' \in Y$, i.e., y maximizes profit given some $p \geq 0$.

- Suppose otherwise that $p \cdot y < p \cdot y''$ for some $y'' \in Y$.
- Then there exists $\varepsilon > 0$ sufficiently small such that $p \cdot (y + \varepsilon e) < p \cdot y''$.
- However, $(y + \varepsilon e) \in P_y$.
- We reach a contradiction with the separation result established in Step 1. $\square$

The end of the second sentence of Proposition 5.F.2 cannot be read as " $p \gg 0$" . The following example illustrates why:

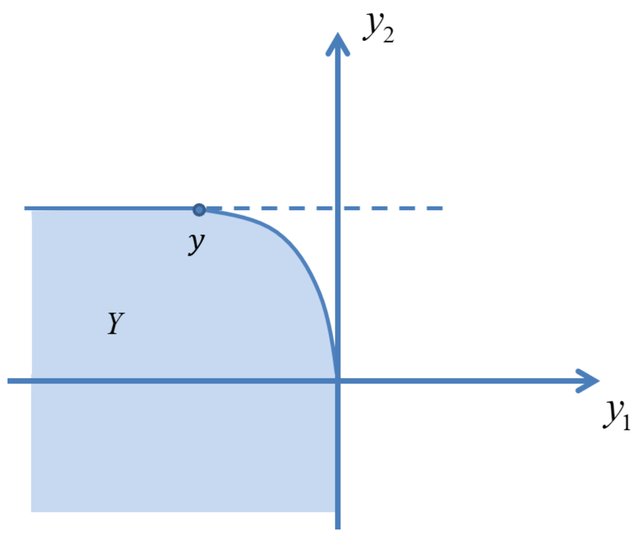

Figure 34: Proposition 5.F.2

The production vector y is efficient but is not profit-maximizing for any $p \gg 0$. It's profit-maximizing for some (p_1, p_2) with $p_1 = 0$.