Chapter 5. Production

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5.A. Introduction

In this chapter, we study the **supply side** of the economy. In particular, we study how goods and services are produced by “firms”.

- We view firms as “black boxes”, transforming inputs into outputs. (simplification)

- The study of organizational structure, witch falls outside of the scope of this chapter, is also important and interesting.

5.B. Production Sets

- We consider an economy with L commodities.

- *Production vector* (**including both inputs & outputs**) $y = (y_1, ..., y_L) \in \mathbb{R}^L$ describes the (net) outputs.

  – If $y_l > 0$, l is an output;

  – If $y_l \leq 0$, l is an input.

Production Sets

Example 5.B.1. Suppose that $L = 5$, Then $y = (-5, 2, -6, 3, 0)$ means that

(a) 2 and 3 units of Good 2 and 4 are produced;

(b) 5 and 6 units of Good 1 and 3 are used;

(c) Good 5 is neither produced or used.

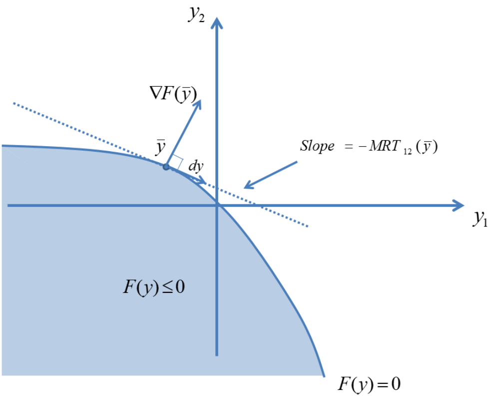
Production Sets

- The set of all production vectors that constitute technologically feasible plans is called the *production set* $Y \subset \mathbb{R}^L$.
  - Any $y \in Y$ is feasible;
  - Any $y \notin Y$ is not feasible.

Production Sets

- We can describe the production set Y by a *transformation function* $F(\cdot)$.
  - The production set $Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$.
  - $\{y \in \mathbb{R}^L : F(y) = 0\}$ is called the *transformation frontier*.

Production Sets



Production Function and Transformation Frontier

Production Sets

- Consider changes in y while staying on $F(y) = 0$.

  For such changes dy along the frontier, we have

$$dy \cdot \nabla F(y) = 0.$$

- Suppose only y_l & y_k change.

$$\frac{dy_k}{dy_l} = -\frac{\partial F(\bar{y})/\partial y_l}{\partial F(\bar{y})/\partial y_k} = -MRT_{lk}(\bar{y}).$$

  $MRT_{lk}(\bar{y})$ is called the *marginal rate of transformation (MRT) of good l for good k at $\bar{y}$.*

Technologies with Distinct Inputs and Outputs

- Suppose there are M outputs and $L - M$ inputs.
  - let $q = (q_1, ..., q_M) \geq 0$ denote the outputs.
  - let $z = (z_1, ..., z_{L-M}) \geq 0$ denote the inputs.
  - e.g. $(y_{L-M+1}, ..., y_L) = (q_1, ..., q_M)$;

    $(y_1, ..., y_{L-M}) = -(z_1, ..., z_{L-M})$.

Technologies with Distinct Inputs and Outputs

- Single-output technology
  - *Production function*: $f(z)$, where

$$z = (z_1, ..., z_{L-1}) \geq 0$$

  - Output: $q \leq f(z)$
  - Production set:

$$Y = \{(-z_1, ..., -z_{L-1}, q) : q - f(z_1, ..., z_{L-1}) \leq 0 \text{ and } (z_1, ..., z_{L-1}) \geq 0\}$$

Technologies with Distinct Inputs and Outputs

- Holding the level of output fixed, we define *Marginal rate of technological substitution (MRTS) of input l for input k at $\bar{z}$* as follows:

$$MRTS_{lk}(\bar{z}) = \frac{\partial f(\bar{z})/\partial z_l}{\partial f(\bar{z})/\partial z_k}$$

  - $MRTS_{lk}(\bar{z})$ is the same as $MRT_{lk}(\bar{z}, \bar{q})$, simply a renaming for the substitution between inputs in a single-output case.

Example 5.B.2. Cobb-Douglas Production Function:

$$f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}, \text{ where } a \geq 0, \beta \geq 0.$$

MRTS at $z = (z_1, z_2)$ is

$$MRTS_{12}(z) = \frac{\partial f(z_1, z_2)/\partial z_1}{\partial f(z_1, z_2)/\partial z_2} = \frac{\alpha z_1^{\alpha-1} z_2^{\beta}}{\beta z_1^{\alpha} z_2^{\beta-1}} = \frac{\alpha z_2}{\beta z_1}.$$

Remark. In percentage change terms

$$\left[\frac{\partial f(z_1, z_2)}{\partial z_1} \frac{z_1}{f(z_1, z_2)}\right] \Bigg/ \left[\frac{\partial f(z_1, z_2)}{\partial z_2} \frac{z_2}{f(z_1, z_2)}\right] = \frac{\alpha z_2}{\beta z_1} \frac{z_1}{z_2} = \frac{\alpha}{\beta}.$$

Commonly Assumed Properties of Production Sets

(i) Y is nonempty.

(ii) Y is closed. (technical)

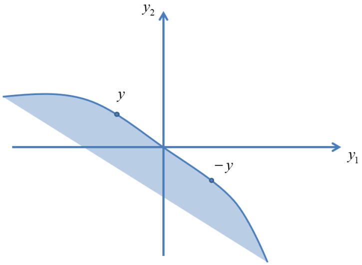
(iii) No free lunch: If $y \geq 0$, $y = 0$. The idea is that no commodities can be created out of thin air. Production of any commodity requires consumption of some other commodities.

(iv) Possibility of inaction: $0 \in Y$.

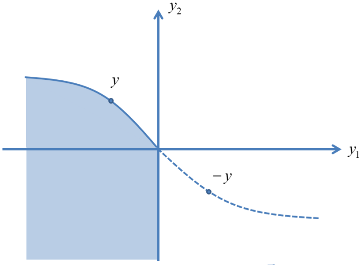
(v) Free disposal: If $y \in Y$ and $y' \leq y$, then $y' \in Y$.

Commonly Assumed Properties of Production Sets

(vi) Irreversibility: Suppose $y \in Y$ and $y \neq 0$, then $-y \notin Y$.



Reversible Technology

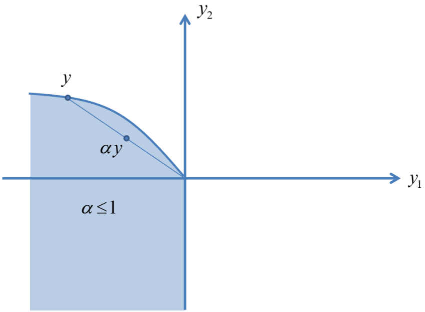


Irreversible Technology

Commonly Assumed Properties of Production Sets

(vii) Nonincreasing returns to scale:

$$y \in Y \text{ and } \alpha \in [0,1] \implies \alpha y \in Y.$$

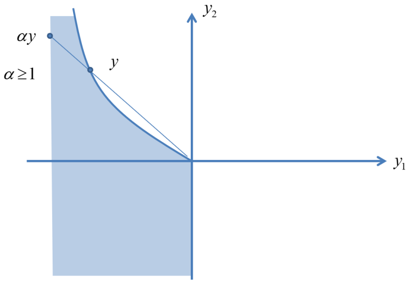


Nonincreasing Returns to Scale Technology

Commonly Assumed Properties of Production Sets

(viii) Nondecreasing returns to scale:

$$y \in Y \text{ and } \alpha \geq 1 \implies \alpha y \in Y.$$

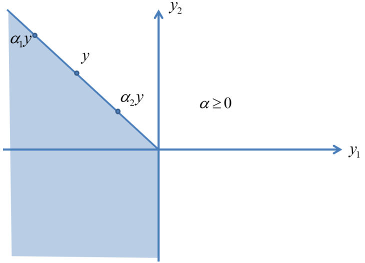


Nondecreasing Returns to Scale Technology

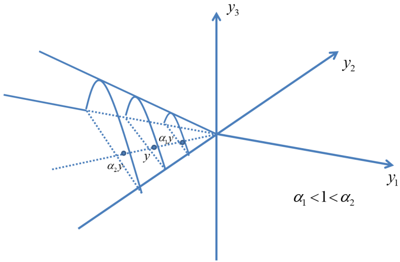
Commonly Assumed Properties of Production Sets

(ix) Constant returns to scale (Cone):

$$y \in Y \text{ and } \alpha \geq 0 \implies \alpha y \in Y.$$



CRS (2 commodities)



CRS (3 commodities)

Commonly Assumed Properties of Production Sets

(x) Additivity: Suppose $y \in Y$ and $y' \in Y$. Then $y + y' \in Y$.

- Alternatively, $Y + Y \subset Y$.
- If $y \in Y$, then $ky \in Y$ for all $k \in \mathbb{Z}$.
- This captures an economy with **free entry**. Any existing technology can be added to the existing technologies.

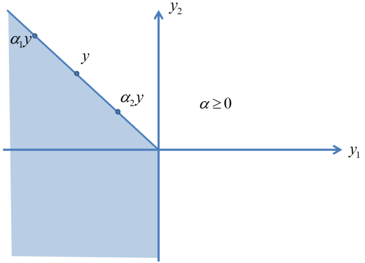
Commonly Assumed Properties of Production Sets

(xi) Convexity:

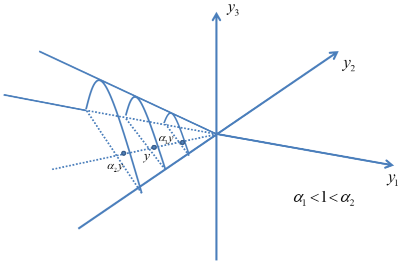
$$y, y' \in Y \text{ and } \alpha \in [0,1] \implies \alpha y + (1-\alpha) y' \in Y.$$

Commonly Assumed Properties of Production Sets

(xii) Convex cone: Y is a convex cone if for any production vector $y, y' \in Y$ and constants $\alpha \geq 0$ & $\beta \geq 0$, we have $\alpha y + \beta y' \in Y$.



Convex Cone (2 commodities)

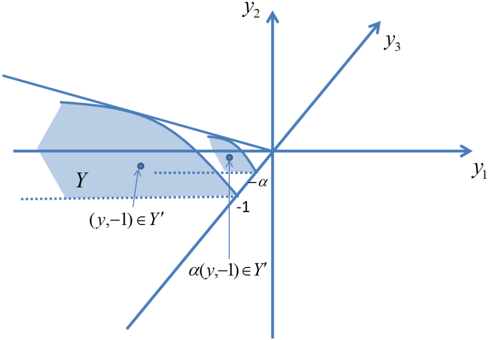


Convex Cone (3 commodities)

Commonly Assumed Properties of Production Sets

Proposition 5.B.1. *The production set Y is additive and satisfies the nonincreasing returns condition if and only if it is a convex cone.*

Proposition 5.B.2. *For any convex production set* $Y \subset \mathbb{R}^L$ *with* $0 \in Y$*, there is a constant returns, convex production set* $Y' \subset \mathbb{R}^{L+1}$ *s.t.* $Y = \{y \in \mathbb{R}^L : (y, -1) \in Y'\}$.



Commonly Assumed Properties of Production Sets

Remark. In essence, the implication is that in a competitive, convex setting, there may be little loss of conceptual generality in limiting to constant returns technologies.

5.C. Profit Maximization and Cost Minimization

- L commodities, priced at $p = (p_1, ...p_L) \gg 0$.

- Firm is *price-taking*.

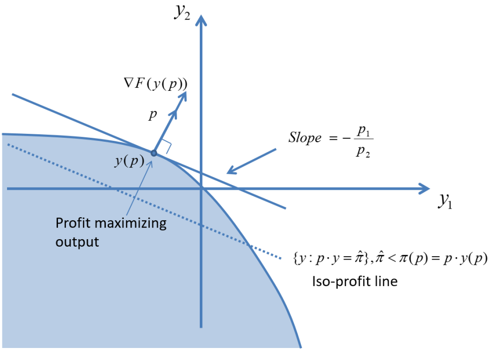
- Firm's objective is to maximize profit.

- Assume *nonemptiness*, *closedness*, and *free disposal*.

Profit Maximization Problem

$$\max_{y} p \cdot y$$

$$\text{s.t. } y \in Y(\text{ or } F(y) \leq 0)$$



Profit Maximization Problem

- Lagrange Function: $\mathcal{L} = p \cdot y - \lambda F(y)$

- Kuhn-Tucker Conditions:[1]

$$\frac{\partial \mathcal{L}}{\partial y_l} = p_l - \lambda \frac{\partial F(y)}{\partial y_l} = 0 \text{ for } l = 1, .., L, \tag{1}$$

$$\lambda \geq 0$$

$$\lambda F(y) = 0$$

$$F(y) \leq 0$$

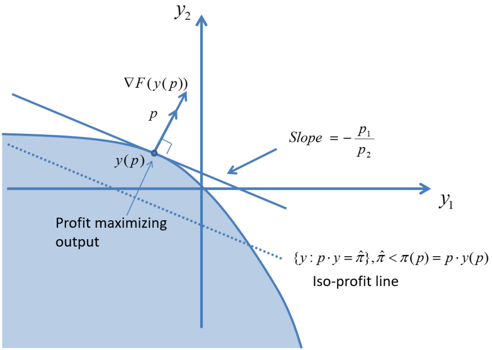
[1] Suppose $F(\cdot)$ is differentiable.

Profit Maximization Problem

Claim. $F(y) = 0$.

Profit Maximization Problem

Remark. Equation (1) implies $\frac{p_l}{p_k} = \frac{\partial F(y^*)/\partial y_l}{\partial F(y^*)/\partial y_k} = MRT_{lk}(y^*)$.



Profit Maximization Problem: Single-output Production

- The profit maximization problem is

$$\max_{z \geq 0, q \geq 0} pq - w \cdot z$$

$$\text{s.t. } q \leq f(z)$$

- The above profit maximization problem could equivalently be written as

$$\max_{z \geq 0} pf(z) - w \cdot z$$

Profit Maximization Problem: Single-output Production

- Lagrange Function: $\mathcal{L} = pf(z) - w \cdot z$

- Kuhn-Tucker Conditions:

$$p\frac{\partial f(z^*)}{\partial z_l} - w_l \leq 0, \quad \text{with equality if } z_l^* > 0 \qquad (2)$$

$$z^* \geq 0$$

Profit Maximization Problem: Single-output Production

- Equation (2) is equivalent to

  $p\nabla f(z^*) \le w$ and $[p\nabla f(z^*) - w] \cdot z^* = 0.$

- Suppose $(z_l^*, z_k^*) \gg 0$. Then,

$$\frac{w_l}{w_k} = \frac{\partial f(z^*)/\partial z_l}{\partial f(z^*)/\partial z_k} = MRTS_{lk}(z^*). \tag{3}$$

- Condition (3) can also be rewritten as

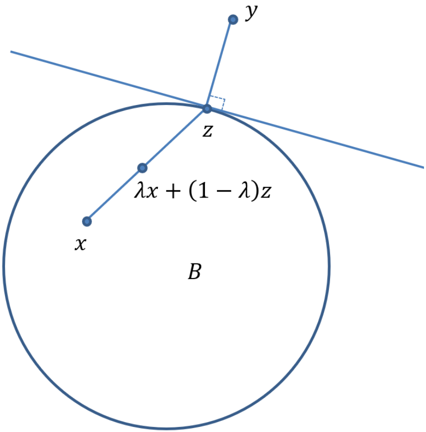
$$\frac{1}{w_l}\frac{\partial f(z^*)}{\partial z_l} = \frac{1}{w_k}\frac{\partial f(z^*)}{\partial z_k} = \text{ marginal product of \$1.}$$

Profit Maximization Problem

If the production set Y is convex, then the F.O.C in (1) and (2) are not only necessary but also sufficient.

Theorem M.G.2 (Separating Hyperplane Theorem (Part I)). *Suppose that* $\mathcal{B} \subset \mathbb{R}^N$ *is convex and closed, and that* $y \notin \mathcal{B}$. *Then there is a* $p \in \mathbb{R}^N$ *with* $p \neq 0$, *and a value* $c \in \mathbb{R}$ *such that* $p \cdot y > c$ *and* $p \cdot x < c$ *for every* $x \in \mathcal{B}$.

Mathematical Appendix: Separating Hyperplane Theorem



Seperating Hyperplane

Proposition 5.C.1. *Suppose $\pi(\cdot)$ is the profit function of the production set Y and that $y(\cdot)$ is the associated supply correspondence. Assume also that Y is closed and satisfies the free disposal property. Then,*

(i) *$\pi(\cdot)$ is homogeneous of degree one.*

(ii) *$\pi(\cdot)$ is convex.*

(iii) *If Y is convex, then $Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p)$ for all $p \gg 0\}$.*

Proposition 3.C.1 (continued).

(iv) $y(\cdot)$ *is homogeneous of degree zero.*

(v) *If* Y *is convex, then* $y(p)$ *is a convex set for all* p. *Moreover, if* Y *is strictly convex, then* $y(p)$ *is single-valued (if nonempty).*

(vi) *(Hotelling's lemma) If* $y(\bar{p})$ *consists of a single point, then* $\pi(\cdot)$ *is differentiable at* $\bar{p}$ *and* $\nabla\pi(\bar{p}) = y(\bar{p})$.

Proposition 3.C.1 (continued).

(vii) *If* $y(\cdot)$ *is a function differentiable at* $\bar{p}$, *then* $Dy(\bar{p}) = D^2\pi(\bar{p})$ *is a symmetric and positive semidefinite matrix with* $Dy(\bar{p})\bar{p} = 0$.

Remark. $\nexists$ budget constraint, so no "income" effect associated with price change.

Law of Supply

Claim. $(p-p')\cdot(y-y') \geq 0$ [That is, $dp\cdot dy = dp^T Dy dp \geq 0$]

Cost Minimization Problem

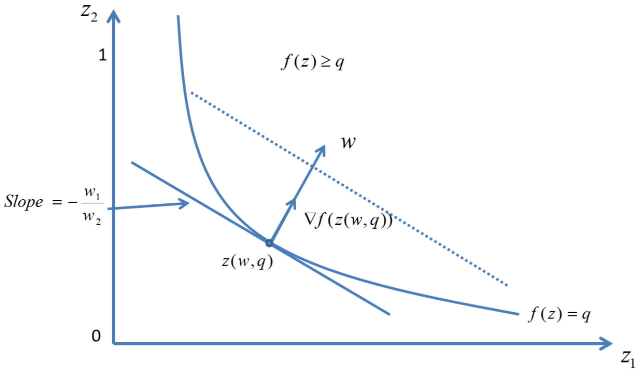
- Cost minimization is necessary (but not sufficient) for profit maximization.

- We focus on single-output production.

- Cost Minimization Problem (CMP):

$$\begin{array}{ccc} \min\limits_{z \geq 0} w \cdot z & & \max\limits_{z \geq 0} \; -w \cdot z \\ & \equiv & \\ \text{s.t. } f(z) \geq q & & \text{s.t. } -f(z) \leq -q \end{array}$$

Cost Minimization Problem



CMP for Single-output Production

Cost Minimization Problem

- $z(w, q)$: solution of CMP
  - $z(w, q)$ is known as the *conditional factor demand function* or correspondence.
- $c(w, q)$: minimized cost, or the cost function.

Cost Minimization Problem

- Lagrange Function: $\mathcal{L} = (-w \cdot z) - \lambda(-f(z) + q)$

- Kuhn-Tucker Conditions:

$$-w_l + \lambda \frac{\partial f(z^*)}{\partial z_l} \leq 0, \text{with equality if } z_l^* > 0 \qquad (4)$$

$$\lambda \geq 0$$

$$\lambda(-f(z) + q) = 0$$

$$-f(z) \leq -q$$

$$z \geq 0$$

Cost Minimization Problem

- Equation (4) is equivalent to $w \geq \lambda \nabla f(z^*)$ and $[w - \lambda \nabla f(z^*)] \cdot z^* = 0$.

- For any l, k with $(z_l, z_k) \gg 0$, we have

$$\frac{w_l}{w_k} = \frac{\partial f(z^*)/\partial z_l}{\partial f(z^*)/\partial z_k} = MRTS_{lk}$$

- λ measures $\partial c(w, q)/\partial q$, or the marginal cost of production.

Cost Minimization Problem

As with Profit Maximization Problem, if the production set Y is convex, then F.O.C. (Equation (4)) is not only necessary but also sufficient for z^* to be an optimum in Cost Minimization Problem.

Proposition 5.C.2. *Suppose that $c(w, q)$ is the cost function and that $z(w, q)$ is the associated conditional factor demand correspondence. Assume also that Y is closed and satisfies the free disposal property. Then,*

(i) $c(\cdot)$ *is H.D.1 in w and nondecreasing in q.*

(ii) $c(\cdot)$ *is a concave function of w.*

(iii) If the sets $\{z \geq 0 : f(z) \geq q\}$ are convex for every q, then $Y = \{(-z, q) : w \cdot z \geq c(w, q)$ for all $w \gg 0\}$.

Cost Minimization Problem

Proposition 5.C.2 (continued).

(iv) $z(\cdot)$ *is homogeneous of degree zero in* w.

(v) *If the set* $\{z \geq 0 : f(z) \geq q\}$ *is convex, then* $z(w, q)$ *is a convex set. Moreover, if* $\{z \geq 0 : f(z) \geq q\}$ *is a strictly convex set, then* $z(w, q)$ *is single-valued.*

(vi) *(Shepard's lemma) If* $z(\bar{w}, q)$ *consists of a single point, then* $c(\cdot)$ *is differentiable with respect to* w *at* $\bar{w}$ *and* $\nabla_w c(\bar{w}, q) = z(\bar{w}, q)$.

Cost Minimization Problem

Proposition 5.C.2 (continued).

(vii) *If* $z(\cdot)$ *is differentiable at* $\bar{w}$*, then* $D_w z(\bar{w}, q) = D_w^2 c(\bar{w}, q)$ *is symmetric and NSD matrix with* $D_w z(\bar{w}, q)\bar{w} = 0$.

(viii) *If* $f(\cdot)$ *is H.D.1 (i.e., exhibits constant returns to scales), then* $c(\cdot)$ *and* $z(\cdot)$ *are H.D.1 in* q.

(ix) *If* $f(\cdot)$ *is concave, then* $c(\cdot)$ *is a convex function of* q *(in particular, marginal costs are nondecreasing in* q*).*

Cost Minimization Problem

Remark. Note that cost minimization is very similar to expenditure minimization.

From Cost Minimization to Profit Maximization

Restate Profit Maximization Problem using the cost function:

$$\max_{q \geq 0} \; pq - c(w, q).$$

Kuhn-Tucker Conditions:

$$p - \frac{\partial c(w, q^*)}{\partial q} \leq 0 \text{ with equality if } q^* > 0 \tag{5}$$

$$q \geq 0.$$

From Cost Minimization to Profit Maximization

- Equation (5) indicates that at an interior optimum (i.e., if $q^* > 0$), *price equals marginal cost.*

- If $c(w, q)$ is convex in q, then the F.O.C (Equation (5)) is not only necessary but also sufficient for q^* to be the optimal production level.

Example 5.C.1. (Building on Example 5.B.2): Derive the cost and profit functions for the Cobb-Douglas production function $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$.

Remark. Since

$$f(\lambda z_1, \lambda z_2) = \lambda^{\alpha+\beta} z_1^{\alpha} z_2^{\beta}$$

Note that $f(\cdot)$ is constant returns to scale if $\alpha + \beta = 1$, increasing returns to scale if $\alpha + \beta > 1$, and decreasing returns to scale if $\alpha + \beta < 1$.

5.D. The Geometry of Cost and Supply on the Single-Output Case

Focusing on the single-output case, we analyze the relationships among: technology, cost function, and supply behavior. We consider fixed factor prices $\bar{w} \gg 0$, and suppress the dependence on w, defining

$$C(q) = c(\bar{w}, q)$$

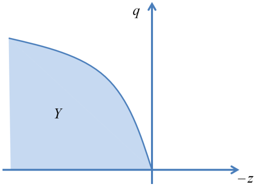
$$AC(q) = c(\bar{w}, q)/q$$

$$MC(q) = \partial c(\bar{w}, q)/\partial q$$

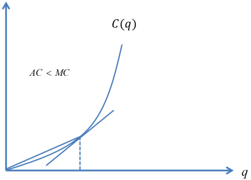
Convex Production Set

Recall F.O.C for profit maximization: $p \leq C'(q)$ with equality if $q > 0$.

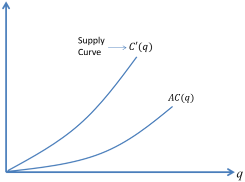
If Y is convex, $c(\cdot)$ is convex and F.O.C is sufficient for profit maximization.



Production Set



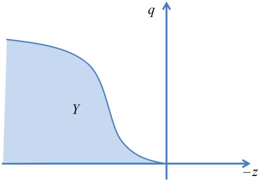
Cost Function



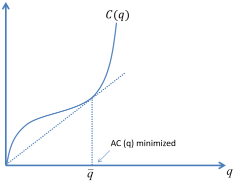
MC and AC

Nonconvex Production Set

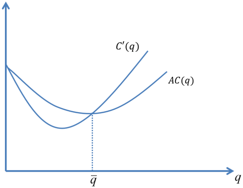
Y may not be convex.



Production Set



Cost Function

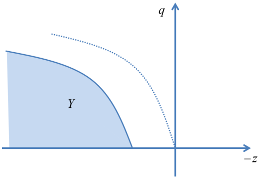


MC and AC

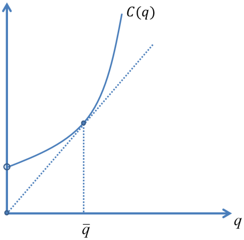
Remark. AC is minimized when $MC(\bar{q}) = AC(\bar{q})$.

Fixed cost (but not sunk)

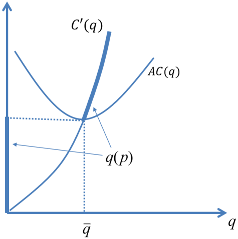
Some input(s) have to be used before any output can be produced. Fix cost is preventable. For active firms, the price has to at least cover the average cost of production.



Production Set



Cost Function



MC and AC

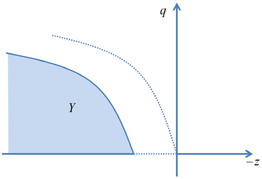
Sunk cost

- When cost is sunk, it is no longer preventable. So it is not an option to use no inputs and incur no cost.

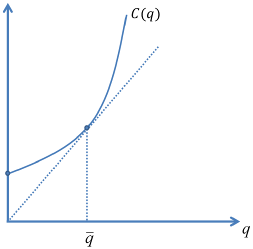
- In deciding whether to be active in production or not, sunk cost should not be part of the consideration because by gone is by gone.

- Therefore, even if the price falls below the average cost, it may still be economically profitable to be active in production.

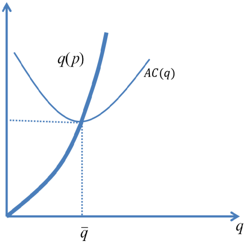
Sunk cost



Production Set

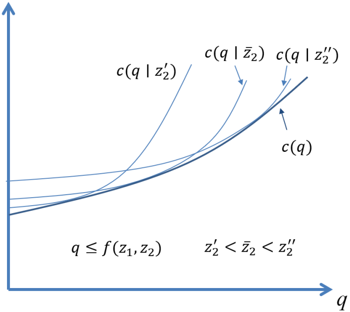


Cost Function

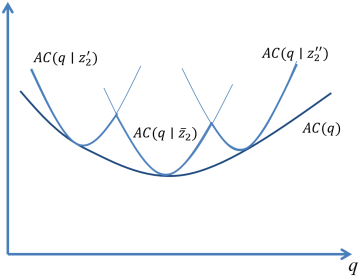


MC and AC

Long-run and short-run cost functions



LR and SR Cost Functions



LR and SR AC

5.E. Aggregation

Question. Would the properties of individual supplies be preserved when they are aggregated to market supply?

Question. Would merger affect supply behavior?

Aggregation

- J production units/plants
- Y_j is nonempty, closed
- $\pi_j(p)$: profit function
- $y_j(p)$: supply correspondence
- Aggregate supply correspondence:

$$y(p) = \sum_{j=1}^{J} y_j(p) = \{y \in \mathbb{R}^L : y = \sum_j y_j \text{ for some } y_j \in y_j(p)\}$$

Aggregation

- Suppose $y_j(p)$ is single-valued & differentiable.
  - $Dy(p) = \sum_j Dy_j(p)$ is also symmetric and PSD.
  - PSD implies law of supply in aggregate:

$$dp \cdot dy = dp^T Dy(p) dp \geq 0.$$

  - Alternatively, from $(p - p') \cdot [y_j(p) - y_j(p')] \geq 0$,

$$(p - p') \cdot \left[\sum_j y_j(p) - \sum_j y_j(p') \right] \geq 0.$$

Aggregation

Aggregate production set:

$$
\begin{aligned}
Y &= Y_1 + ... + Y_J \\
&= \{y \in \mathbb{R}^L : y = \sum_j y_j \text{ for some } y_j \in Y_j, j = 1, ..., J\}
\end{aligned}
$$

- Y is feasible to a single owner who maximizes total profit from J plants' production.
- $\pi^*(p)$ and $y^*(p)$ are the profit function and the supply correspondence of Y.

Aggregation

Proposition 5.E.1. *For all* $p \gg 0$, *we have*

(i) $\pi^*(p) = \sum_j \pi_j(p)$

(ii) $y^*(p) = \sum_j y_j(p)$

Remark. This result that merger does affect supply behavior holds only because the firms are **price takers**.

5.F. Efficient Production (Narrow notion of efficiency)

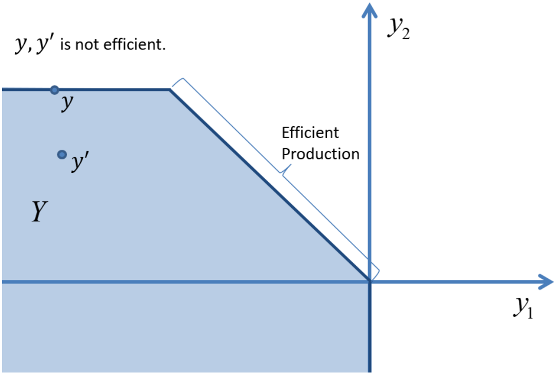
Question. When do we regard production as nonwasteful?

Efficient Production

We take the prices as exogenously fixed and do not discuss whether the prices are too high or too low when we discuss the efficiency of a profit maximizing firm.

Definition 5.F.1. A production vector $y \in Y$ is efficient if there is no $y' \in Y$ such that $y' \geq y$ and $y' \neq y$.

Efficient Production



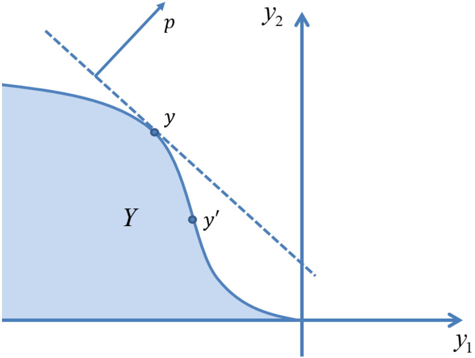
(In)Efficient Production

Efficient Production

Proposition 5.F.1. *If* $y \in Y$ *is profit maximizing for some* $p \gg 0$, *then* y *is efficient.*

Efficient Production

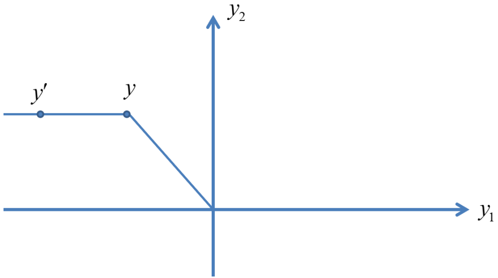
Remark. Proposition 5.F.1 holds even when the production set is non-convex.



Non-convex production set

Efficient Production

Exercise 5.F.1. Suppose $p_1 = 0$ & $p_2 > 0$. Then for all p_2, both y and y' maximize profit but y' is NOT efficient. This illustrates the importance of $p \gg 0$ in Proposition 5.F.1.



$p_1 = 0$

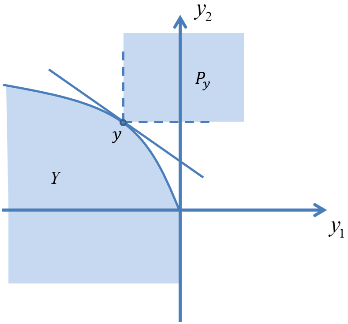
Mathematical Appendix: Supporting Hyperplane Theorem

Theorem M.G.3 (Supporting Hyperplane Theorem)**.** *Suppose that* $\mathcal{B} \subset \mathbb{R}^N$ *is convex and that* x *is not an element of the interior of set* $\mathcal{B}$ *(*$x \notin \operatorname{Int} \mathcal{B}$*). Then there is* $p \in \mathbb{R}^N$ *with* $p \neq 0$ *such that* $p \cdot x \geq p \cdot y$ *for every* $y \in \mathcal{B}$*.*

Theorem M.G.2 (Separating Hyperplane Theorem (Part II)). *Suppose that the convex sets* $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^N$ *are disjoint (i.e.,* $\mathcal{A} \cap \mathcal{B} = \varnothing$*). Then there is* $p \in \mathbb{R}^N$ *with* $p \neq 0$*, and a value* $c \in \mathbb{R}$*, such that* $p \cdot x \geq c$ *for every* $x \in \mathcal{A}$ *and* $p \cdot y \leq c$ *for every* $y \in \mathcal{B}$*. That is, there is a hyperplane that separates* $\mathcal{A}$ *and* $\mathcal{B}$*, leaving* $\mathcal{A}$ *and* $\mathcal{B}$ *on different sides of it.*

Efficient Production

Proposition 5.F.2. *Suppose that Y is convex. Then every efficient production $y \in Y$ is a profit-maximizing production for some nonzero price vector $p \geq 0$.*



Efficient Production

The end of the second sentence of Proposition 5.F.2 cannot be read as " $p \gg 0$" . The following example illustrates why: