Chapter 8. Second-Order Conditions

Xiaoxiao Hu

March 29, 2022

8.A. Local and Global Maxima

- In Chapter 7, we have discussed sufficient conditions for optimality, confined to context of concave programming (or more broadly, quasi-concave programming).
- Especially, when F is concave and G is convex, FOCs are sufficient for maximization.
- More accurately, the conditions are sufficient for a global maximum.
- That is, x^* satisfying the conditions does at least as well as any other feasible x.

Local and Global Maxima

- We obtain a global maximum in concave programming (quasi-concave programming) since (quasi-)convexity properties are defined globally.
- For example, recall the definition of convexity,

Definition 6.B.4 (Convex Function)**.** A function $f : \mathcal{S} \to \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, is convex if

$$f(\alpha x^a + (1-\alpha)x^b) \leq \alpha f(x^a) + (1-\alpha)f(x^b), \qquad (6.4)$$

for all x^a, $x^b \in \mathcal{S}$ and for all $\alpha \in [0, 1]$.

Local and Global Maxima

- (6.4) needs to hold over the full domain of f.
- Such properties ensure that desired curvature is over the full domain and thus sufficient for a global maximum.

Local and Global Maxima

- The conclusions of a global maximiximum are ideal.

- However, in applications, we may not have functions that have desired convexity property.

Local and Global Maxima

- In this chapter, we will focus on curvature of the objective and constraint functions in a small neighborhood of the proposed optimum.
- The conditions are expressed in terms of second-order derivatives of functions at the point.
- Such conditions are sufficient for local optima – x^* satisfying the conditions does better than any other feasible x in a sufficiently small neighborhood of x^*.

Local and Global Maxima

- It is a useful property when global conditions are not met.
- Moreover, it has a valuable by-product: SOCs play an instrumental role in determining comparative static responses of optimum choice variables x.
- We will discuss comparative static result while we develop the theory of SOCs.

8.B. Unconstrained Maximization

- We will start with simple cases of unconstrained maximization.
- First, consider unconstrained maximization problem with a scalar x:

$$\max_x F(x).$$

- Let x^* be a candidate for optimum choice.

Unconstrained Maximization

- Expand F in a Taylor series around x^*:

$$F(x) = F(x^*) + F'(x^*)(x - x^*) + \frac{1}{2}F''(x^*)(x - x^*)^2 + ... \quad (8.1)$$

- First-order necessary condition is $F'(x^*) = 0$.
- Then (8.1) becomes

$$F(x) - F(x^*) = \frac{1}{2}F''(x^*)(x - x^*)^2 + ... \quad (8.2)$$

- For x sufficiently close to x^*, quadratic term will dominate higher-order terms in Taylor expansion.

Unconstrained Maximization

For x in the small neighborhood of x^*.

(i) $F''(x^*) > 0 \implies F(x) - F(x^*) > 0 \implies F(x) > F(x^*)$.

- x^* will not be a maximum of $F(x)$ in the neighborhood.
- It will not be a maximum over the whole range of F.
- This argument gives a second-order necessary condition for x^* to yield a maximum, local or global:

$$F''(x^*) \leq 0. \tag{8.3}$$

Unconstrained Maximization

(ii) $F''(x^*) < 0 \implies F(x) - F(x^*) < 0 \implies F(x) < F(x^*)$.

- In a small neighborhood of x^*, we will have $F(x^*) > F(x)$, irrespective of signs of higher-order terms.

- Thus,

$$F''(x) < 0 \tag{8.4}$$

  is a second-order sufficient condition for x^* to yield a local maximum.

Unconstrained Maximization

Note the differences between the weak inequality condition

$$F''(x^*) \leq 0 \tag{8.3}$$

and the strict inequality condition

$$F''(x) < 0 \tag{8.4}$$

(i) (8.3) is a necessary condition, while (8.4) is a sufficient condition.

(ii) (8.3) is a condition for both local and global maximum, while (8.4) is a condition only for local maximum.

Unconstrained Maximization

- A local maximum satisfying second-order sufficient condition is called a regular maximum.
- If the maximum is "irregular", that is , if $F''(x) = 0$, then we have to look at higher-order derivatives.

$$F(x) - F(x^*) = \frac{1}{3!}F'''(x^*)(x - x^*)^3 + \frac{1}{4!}F''''(x^*)(x - x^*)^4 + ...$$

- Then, $F'''(x^*) = 0$ is a necessary condition; $F'''(x^*) = 0$ and $F''''(x) < 0$ is a sufficient condition.
- We will focus on the regular maximum.

Comparative Statics

- Now suppose that the problem involves a parameter θ, that is, the objective function is $F(x, \theta)$.
- FOC is

$$F_x(x^*, \theta) = 0. \tag{8.5}$$

(8.5) implicitly defines x^* as a function of θ.

Comparative Statics

- Totally differentiate FOC, we have

$$
\begin{aligned}
& F_{xx}(x^*, \theta)\mathrm{d}x^* + F_{x\theta}(x^*, \theta)\mathrm{d}\theta = 0 \\
\text{or} \quad & \frac{\mathrm{d}x^*}{\mathrm{d}\theta} = -\frac{F_{x\theta}(x^*, \theta)}{F_{xx}(x^*, \theta)}. \qquad (8.6)
\end{aligned}
$$

- At a regular maximum, $F_{xx}(x^*, \theta) < 0$,

  sign of $\mathrm{d}x^*/\mathrm{d}\theta$ is same as sign of $F_{x\theta}(x^*, \theta)$.

An Economic Illustration

- Consider the following revenue maximization problem:

$$\max_x R(x, \theta) \equiv \max_x P(x, \theta) \cdot x,$$

  where x is the output and θ is a shift parameter; $P(x, \theta)$ is the inverse demand curve.

- Suppose $R_\theta(x, \theta) = P_\theta(x, \theta) \cdot x > 0$ for all x.
- That is, an increase in θ shifts the demand and the revenue curves upward.

An Economic Illustration

- By the first-order necessary condition,

$$R_x(x^*, \theta) = P_x(x^*, \theta) \cdot x^* + P(x^*, \theta) = 0. \tag{8.7}$$

- Totally differentiate (8.7), we have

$$\begin{aligned} R_{xx}(x^*, \theta)\mathrm{d}x^* + R_{x\theta}(x^*, \theta)\mathrm{d}\theta &= 0 \\ \Longrightarrow \frac{\mathrm{d}x^*}{\mathrm{d}\theta} &= -\frac{R_{x\theta}(x^*, \theta)}{R_{xx}(x^*, \theta)} \end{aligned} \tag{8.8}$$

- At a regular maximum, we have $R_{xx}(x^*, \theta) < 0$.
- Therefore, sign of $\mathrm{d}x^*/\mathrm{d}\theta$ is same as sign of $R_{x\theta}(x^*, \theta)$.

An Economic Illustration

- Thus, if $R_{x\theta}(x^*, \theta) > 0$, an increase in θ will increase revenue-maximizing output x^*.
- This is true if the increase in θ shifts marginal revenue upward:

$$\frac{\mathrm{d}R_x(x, \theta)}{\mathrm{d}\theta} > 0.$$

An Economic Illustration

- Of course, it is perfectly possible that as $\theta \uparrow$,

  (i) average revenue shifts up: $P_\theta(x,\theta) > 0$;

  (ii) marginal revenue shifts down: $\mathrm{d}R_x(x,\theta)/\mathrm{d}\theta < 0$.

- What is needed is a twist that reduces elasticity of demand ($E_d > 0$). To see this,

$$R_x(x,\theta) = P_x(x,\theta) + P(x,\theta) = P(x,\theta)\left[1 - \frac{1}{E_d}\right].$$

- If marginal revenue does shift down , then a favorable shift of demand will cause output to fall.

More Choice Variables

- Let us turn to the case with a vector of choice variables.
- Now Taylor expansion becomes

$$\begin{aligned} F(x) =& F(x^*) + F_x(x^*)(x - x^*) \\ &+ \frac{1}{2}(x - x^*)^T F_{xx}(x^*)(x - x^*) + ... \\ =& F(x^*) + \sum_{j=1}^{n} \left[F_j(x^*)(x_j - x_j^*) \right] \\ &+ \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} F_{jk}(x_j - x_j^*)(x_k - x_k^*) + ... \end{aligned} \tag{8.9}$$

More Choice Variables

- FOC is $F_x(x^*) = 0$.
- Then (8.9) becomes

$$F(x) - F(x^*) = \frac{1}{2}(x - x^*)^T F_{xx}(x^*)(x - x^*) + ...$$

More Choice Variables

- For x sufficiently close to x^*, quadratic term dominates high-order terms. Therefore,

  (i) $(x-x^*)^T F_{xx}(x^*)(x-x^*) \leq 0$ is second-order necessary condition for x^* to yield a local or global maximum;

  (ii) $(x-x^*)^T F_{xx}(x^*)(x-x^*) < 0$ is second-order sufficient condition for x^* to yield a local maximum.

More Choice Variables

We will next link second-order derivative test with mathematical concepts of Negative (Semi-)Definiteness of matrices.

Negative (Semi-)Definite Matrix

Definition 8.B.1 (Negative Definite)**.** A symmetric $N \times N$ matrix M is negative definite if

$$y^T M y < 0 \tag{8.10}$$

for all non-zero $y \in \mathbb{R}^N$.

Negative (Semi-)Definite Matrix

Definition 8.B.2 (Negative Semi-definite)**.** A symmetric $N \times N$ matrix M is negative semi-definite if

$$y^T M y \leq 0 \tag{8.11}$$

for all $y \in \mathbb{R}^N$.

Negative (Semi-)Definite Matrix

Example 8.B.1. $M = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ is negative definite.

Negative (Semi-)Definite Matrix

Example 8.B.2. $M = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ is negative semi-definite.

Negative (Semi-)Definite Matrix

- Note that a matrix M with all negative entries may not be negative definite.

- Example 8.B.3 illustrates the case where all entries in M is negative whereas M is not negative definite.

Negative (Semi-)Definite Matrix

Example 8.B.3. $M = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}$ is not negative definite.

In particular, for $y = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, we have $y^T M y > 0$.

Positive (Semi-)Definite Matrix

Similarly, we could define positive (semi-)definite matrices analogously.

Definition 8.B.3 (Positive Definite)**.** A symmetric $N \times N$ matrix M is positive definite if

$$y^T M y > 0 \tag{8.12}$$

for all non-zero $y \in \mathbb{R}^N$.

Positive (Semi-)Definite Matrix

Definition 8.B.4 (Positive Semi-definite)**.** A symmetric $N \times N$ matrix M is positive semi-definite if

$$y^T M y \geq 0 \tag{8.13}$$

for all $y \in \mathbb{R}^N$.

Indefinite Matrix

Remark. A matrix that is not positive semi-definite and not negative semi-definite is called indefinite.

Definiteness of Matrices

- There are various ways to check definiteness of matrices.
- In Examples 8.B.1, 8.B.2 and 8.B.3, we have used the definition to check the definiteness.
- Below, we will introduce determinantal test for definiteness.

Principal Minor

Before discussing the general theorem, we need to learn some new concepts.

Definition 8.B.5 (Principal Submatrix and Principal Minor)**.** Let M be a $N \times N$ matrix. A $k \times k$ submatrix of M formed by deleting $n-k$ rows and the same $n-k$ columns of M is called the k^{th} order principal submatrix of M. The determinant of a principal submatrix is called the k^{th} order principal minor of M.

Principal Minor

Example 8.B.4.

For a general 3×3 matrix $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Leading Principal Minor

Definition 8.B.6 (Leading Principal Submatrix and Leading Principal Minor)**.** Let M be a $N \times N$ matrix. The k^{th} order principal submatrix of M obtained by deleting the last $n - k$ rows and columns of M is called the k^{th} order leading principal submatrix of M; and its determinant is called the k^{th} order leading principal minor of M.

Leading Principal Minor

Example 8.B.5. For the 3×3 matrix in Example 8.B.4,

1. The 3^{rd} order leading principal minor is $\det M$;
2. The 2^{nd} order leading principal minor is $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$;
3. The 1^{st} order leading principal minor is $\det \begin{bmatrix} a_{11} \end{bmatrix}$.

Definiteness of Matrices

The following two theorems provide the algorithm for testing definiteness of a symmetric matrix.

Theorem 8.1. Let M be an $N \times N$ symmetric matrix. Then

1. M is positive definite if and only if all its leading principal minors are positive;

2. M is negative definite if and only if all its leading principal minors of odd order are negative; and all its leading principal minors of even order are positive.

Definiteness of Matrices

Theorem 8.2. Let M be an $N \times N$ symmetric matrix. Then

1. M is positive semi-definite if and only if all its principal minors are non-negative;

2. M is negative semi-definite if and only if all its principal minors of odd order are non-positive ; and all its principal minors of even order are non-negative.

Definiteness of Matrices

Remark. Please note that to check semi-definiteness of matrices, we must unfortunately check not only the leading principal minors, but all principal minors.

Definiteness of Matrices

- Returning to our maximization problem.
- We rewrite SOCs using (semi-)definiteness of matrices.

(i) Second-order necessary condition: $F_{xx}(x^*)$ is negative semi-definite;

(ii) Second-order sufficient condition: $F_{xx}(x^*)$ is negative definite.

Remark. F_{xx} is called Hessian Matrix.

Concavity

- We would like to compare and contrast SOCs with the property of concavity, which is defined globally.

Proposition 7.A.1 (Concave Function)**.** A differentiable function $f : \mathcal{S} \to \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, is concave if and only if

$$f_x(x^a)(x^b - x^a) \geq f(x^b) - f(x^a), \tag{7.1}$$

for all $x^a, x^b \in \mathcal{S}$.

Concavity

For twice continusouly differentiable functions, concavity could be interpreted in terms of second-order derivatives.

Theorem 8.3.

- (Twice continuously differentiable) function $f : \mathcal{S} \to \mathbb{R}$ is concave if and only if f_{xx} is negative semi-definite for every $x \in \mathcal{S}$.

- If f_{xx} is negative definite for every $x \in \mathcal{S}$, then the function is strictly concave.

Concavity

(i) Concavity requires F_{xx} to be negative semi-definite for every x;

(ii) Second-order necessary condition only requires F_{xx} to be negative semi-definite for x^*.

Concavity

- This is why SOCs are useful: it is applicable to functions that do not have desired concavity property over their whole domain of definition.
- Of course, on the other hand, if the function do have the concavity property, it will satisfy second-order necessary condition.

Concavity

Remark. To apply SOCs we derived in this chapter, objective function need not be concave (defined globally).

- Objective function only needs to be "concave" at x^*: $F_{xx}(x^*)$ is negative semi-definite.

Comparative Statics

- Similar to scalar variable case, we could derive comparative static result by

  1. totally differentiating FOC;

  2. applying SOC.

- See Example 8.4 Part I for an application.

8.C. Constrained Optimization

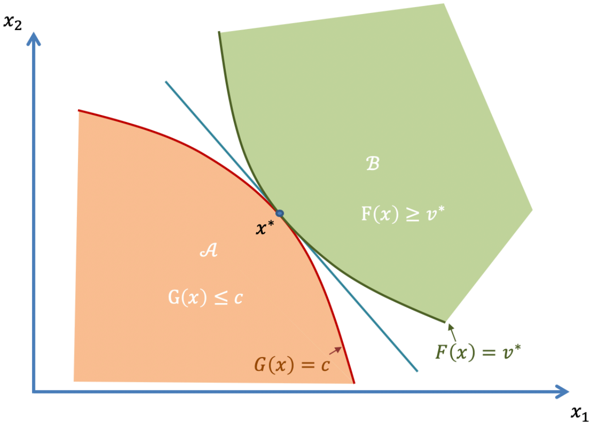
We will begin with the simplest case of two choice variables and one equality constraint.

$$\max_{x_1, x_2} F(x_1, x_2)$$

$$\text{s.t. } G(x_1, x_2) = c$$

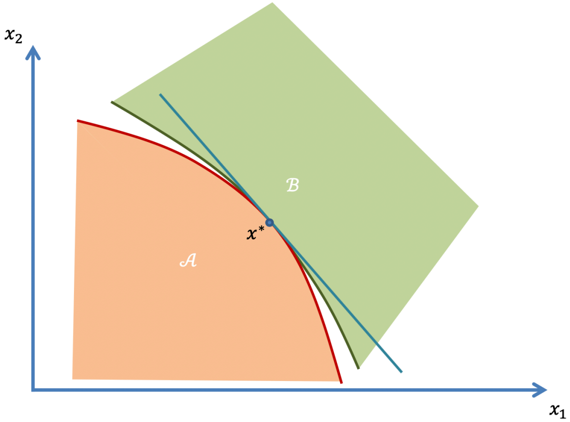
where F and G are increasing functions of their arguments.

Constrained Optimization



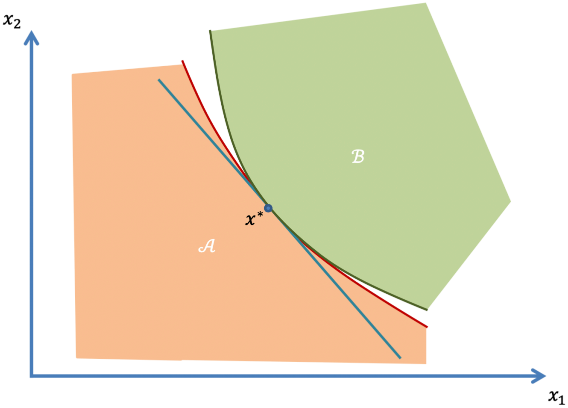
(a)

Constrained Optimization



(b)

Constrained Optimization



(c)

Constrained Optimization

- We have mentioned in Chapter 6 that relative curvature of F and G matters for maximization: contour of F should be more convex than that of G.

- To express the idea algebraically, we think of x_2 as a function of x_1 along contour of F and G, and find second-order derivative of this function.

Constrained Optimization

- For F, function of contour is $F(x_1, x_2) = v$.

- Total differentiation gives

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\frac{F_1(x_1, x_2)}{F_2(x_1, x_2)}. \tag{8.14}$$

- To obtain curvature, differentiate (8.14) with respect x_1:

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}x_1^2} = -\frac{{F_2}^2 F_{11} - 2F_1 F_2 F_{12} + {F_1}^2 F_{22}}{{F_2}^3}.$$

(In the derivation, we used $F_{12} = F_{21}$)

Constrained Optimization

Remark. Symmetry of second derivative matrix follows from Schwarz's theorem: if F has continuous second partial derivative at a, then, $\frac{\partial^2 f(a)}{\partial x_i \partial x_j} = \frac{\partial^2 f(a)}{\partial x_j \partial x_i}$.

Constrained Optimization

A similar expression could be derived for second-order derivative along constraint curve:

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}x_1^2} = -\frac{{G_2}^2 G_{11} - 2G_1 G_2 G_{12} + {G_1}^2 G_{22}}{{G_2}^3}.$$

Constrained Optimization

Second-order sufficient condition for x^* to be a local optimum is that $\mathrm{d}^2 x_2/dx_1{}^2$ along the F contour should be greater than that along the G contour, implying

$$G_2{}^2\,(F_{11} - \lambda G_{11}) - 2G_1G_2(F_{12} - \lambda G_{12}) + G_1{}^2(F_{22} - \lambda G_{22}) < 0$$

evaluated at x^*.

Constrained Optimization

This is more neatly expressed in matrix notation:

$$\det \begin{bmatrix} 0 & -G_1 & -G_2 \\ -G_1 & F_{11} - \lambda G_{11} & F_{12} - \lambda G_{12} \\ -G_2 & F_{21} - \lambda G_{21} & F_{22} - \lambda G_{22} \end{bmatrix} > 0, \qquad (8.15)$$

evaluated at x^*.

Generalization to more variables and more constraints

- Next, we provide without proof conditions for general problem with n choice variables and m equation constraints ($m < n$).
- Similar to matrix notation in (8.15), we form the partitioned matrix:

$$\begin{bmatrix} 0 & -G_x \\ -{G_x}^T & F_{xx} - \lambda G_{xx} \end{bmatrix}, \tag{8.16}$$

evaluated at x^*.

- Top left partition 0 is $m \times m$; bottom right partition $F_{xx} - \lambda G_{xx}$ is $n \times n$; and G_x is $m \times n$.

Generalization to more variables and more constraints

Remark. Matrix

$$\begin{bmatrix} 0 & -G_x \\ -G_x{}^T & F_{xx} - \lambda G_{xx} \end{bmatrix}$$

is called Bordered Hessian Matrix.

Generalization to more variables and more constraints

- For second-order sufficient condition, we need to look at $n-m$ of bordered Hessian's leading principal minors.

- Intuitively, we can think of m constraints as reducing the problem to one with $n-m$ free variables.

- For example, maximization problem:

  $\max_{x,y,z} x + y^2 + z$ subject to $x + y + z = 1$

  can be reduced to

  $\max_{x,y} x + y^2 + (1 - x - y)$ with no constraint.

Generalization to more variables and more constraints

- Smallest minor we consider consisting of truncated first $2m+1$ rows and columns, next consisting of truncated first $2m+2$ rows and columns, and so on, with last being determinant of entire bordered Hessian.

- A sufficient condition for a local maximum of F is that smallest minor has same sign as $(-1)^{m+1}$ and that rest of the principal minors alternate in sign.

Generalization to more variables and more constraints

Theorem 8.4 (Second-order Sufficient Condition for Constrained Maximization Problem). If the last $n - m$ leading principal minors of bordered Hessian matrix at proposed optimum x^* is such that

- smallest minor ($(2m+1)^{th}$ minor) has same sign as $(-1)^{m+1}$,
- rest of principal minors alternate in sign,

then x^* is local maximum.

Generalization to more variables and more constraints

It is easy to check that (8.15) satisfies the sufficient condition for a local maximum for two-variable one-constraint case:

1. For two-variable one-constraint case ($n = 2, m = 1$), we need to look at $n-m = 1$ leading principal minors. Therefore, we only need to compute determinant of bordered Hessian.

2. Sign requirement for maximum is $(-1)^{m+1} = (-1)^2 > 0$.

Generalization to more variables and more constraints

Example 8.C.1. Consider the maximization problem with three variables ($n = 3$) and two constraints ($m = 2$):

$$\max_{x,y,z} F(x, y, z) \equiv z$$

$$\text{s.t. } G^1(x, y, z) \equiv x + y + z = 12$$

$$G^2(x, y, z) \equiv x^2 + y^2 - z = 0$$

More variables and more constraints: Example 8.C.1

- Stationary points are $(x^*, y^*, z^*, \lambda, \mu) = (2, 2, 8, \frac{4}{5}, -\frac{1}{5})$ and $(-3, -3, 18, \frac{6}{5}, \frac{1}{5})$.
- Bordered Hessian matrix is

$$\begin{bmatrix} 0 & 0 & -G_x^1 & -G_y^1 & -G_z^1 \\ 0 & 0 & -G_x^2 & -G_y^2 & -G_z^2 \\ -G_x^1 & -G_x^2 & \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ -G_y^1 & -G_y^2 & \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ -G_z^1 & -G_z^2 & \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & -2x & -2y & 1 \\ -1 & -2x & -2\mu & 0 & 0 \\ -1 & -2y & 0 & -2\mu & 0 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

More variables and more constraints: Example 8.C.1

- We need to check $n - m = 1$ leading principal minors, i.e., we only need to check determinant of the bordered Hessian.
- Sign requirement for maximum is $(-1)^{m+1} = (-1)^3 < 0$.

More variables and more constraints: Example 8.C.1

1. 1^{st} proposed optimum: $(x^*, y^*, z^*, \lambda, \mu) = (2, 2, 8, \frac{4}{5}, -\frac{1}{5})$
   - Determinant of bordered Hessian is $20 > 0$.
2. 2^{nd} proposed optimum: $(x^*, y^*, z^*, \lambda, \mu) = (-3, -3, 18, \frac{6}{5}, \frac{1}{5})$
   - Determinant of bordered Hessian is $-20 < 0$.

Therefore, 2^{nd} proposed optimum $(x^*, y^*, z^*, \lambda, \mu) = (-3, -3, 18, \frac{6}{5}, \frac{1}{5})$ is a local maximum.

Generalization to more variables and more constraints

Example 8.C.2. Consider the following maximization problem with three variables ($n = 3$) and one constraint ($m = 1$):

$$\max_{x,y,z} F(x, y, z) \equiv x + y + z$$

$$\text{s.t. } G^1(x, y, z) \equiv x^2 + y^2 + z^2 = 3$$

More variables and more constraints: Example 8.C.2

- Stationary points are $(x^*, y^*, z^*, \lambda) = (-1, -1, -1, -\frac{1}{2})$ and $(1, 1, 1, \frac{1}{2})$.

- Bordered Hessian matrix is

$$\begin{bmatrix} 0 & -G_x^1 & -G_y^1 & -G_z^1 \\ -G_x^1 & \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ -G_y^1 & \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ -G_z^1 & \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix} = \begin{bmatrix} 0 & -2x & -2y & -2z \\ -2x & -2\lambda & 0 & 0 \\ -2y & 0 & -2\lambda & 0 \\ -2z & 0 & 0 & -2\lambda \end{bmatrix}$$

More variables and more constraints: Example 8.C.2

- We need to check $n - m = 2$ leading principal minors, i.e., 3^{rd} order and entire bordered Hessian.
- For local maximum, sign requirement is
  1. $(-1)^{m+1} = (-1)^2 > 0$ for the 3^{rd} order leading principal minor and
  2. < 0 for entire bordered Hessian.

More variables and more constraints: Example 8.C.2

1. First proposed optimum: $(x^*, y^*, z^*, \lambda) = (-1, -1, -1, -\frac{1}{2})$
   - 3^{rd} order leading principal minor is $-8 < 0$;
   - Determinant of bordered Hessian is $-12 < 0$.
2. Second proposed optimum: $(x^*, y^*, z^*, \lambda) = (1, 1, 1, \frac{1}{2})$
   - 3^{rd} order leading principal minor is $8 > 0$;
   - Determinant of bordered Hessian is $-12 < 0$.

Thus, 2^{nd} proposed optimum $(x^*, y^*, z^*, \lambda) = (1, 1, 1, \frac{1}{2})$ is a local maximum.

Comparative Statics

- For constrained maximization problem, we could derive comparative static results by

  (i) totally differentiating FOCs and constrained equations;

  (ii) applying SOCs.

- See Example 8.4 Part II for an application.

Inequality Constraints

Finally, we consider maximization problem

$$\max_x F(x)$$
$$\text{s.t. } G(x) \le c.$$

- After applying Kuhn-Tucker first-order necessary conditions and solving for stationary points, we know which constraints are binding and which are not in those candidate optima.

Inequality Constraints

- It seems that for each stationary point, we could treat binding constraints as equality constraints and simply ignore slack constraints.
- The intuition is correct in general, but there is one tricky point: it is possible that inequality constraint is binding but at the same time its corresponding Lagrange multiplier is equal to 0.
- These inequality constraints are degenerate inequality constraints.

Inequality Constraints

- Conclusion is that to check second-order sufficient condition, we should only keep binding constraints with strictly positive corresponding Lagrange multipliers.

- In other words, we form bordered Hessian Matrix using only constraints with strictly positive Lagrange multipliers and then apply Theorem 8.4.

8.D. Envelope Properties

In Chapter 5, we established envelope property of maximum value function:

$$V(\theta) = \max_x \{F(x, \theta) \mid G(x) \leq c\}.$$

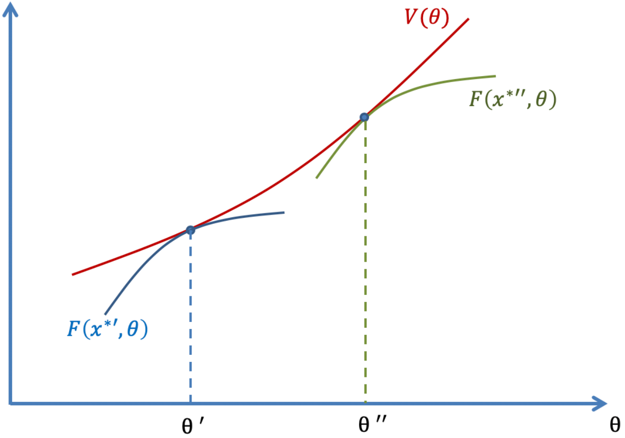
- $V(\theta)$ is upper envelope of family of functions $F(x, \theta)$ in each of which x is held fixed.

Envelope Properties

- Subsequently, we have considered more general problem of short-run and long-run maximum value functions, where vector of choice variables x is partitioned into subvectors (y, z) and z is held fixed in short-run.

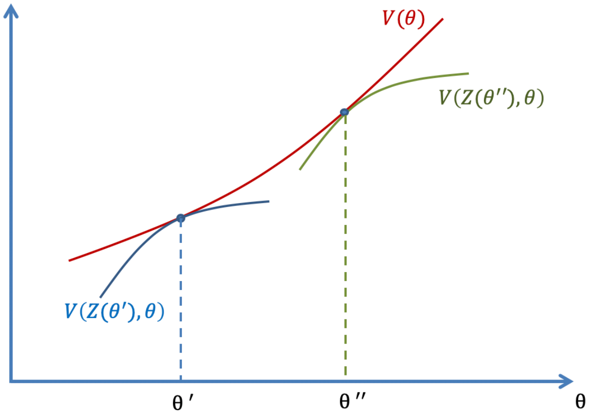
- $V(\theta)$, long-run optimum value function, is upper envelope of family of value functions $V(z, \theta)$, short-run maximum value functions.

Envelope Properties



(a) Envelope Theorem

Envelope Properties



(b) Short-run and Long-run Curves

Envelope Properties

- We have also mentioned curvature properties of envelopes.
- In Figure (a), V is more convex than each F.
- In Figure (b), $V(\theta)$ is more convex than $V(z, \theta)$.
- That is, the fewer variables are held fixed, the more convex should the maximum value function be.
- Second-order envelope property is the subject of this section.

Envelope Properties

- Following same notation of Chapter 5, let $Z(\theta)$ be long-run optimum value of z.
- Then, long-run and short-run value coincide at $Z(\theta)$:

$$V(\theta) = V(Z(\theta), \theta). \tag{8.17}$$

- Besides, two curves are tangential at $Z(\theta)$:

$$V_\theta(\theta) = V_\theta(Z(\theta), \theta). \tag{8.18}$$

Envelope Properties

- Now consider a deviation from θ to θ', we have

$$V(Z(\theta), \theta') \leq V(Z(\theta'), \theta') = V(\theta').$$

- Expand $V(Z(\theta), \theta')$ and $V(\theta')$ around θ in Taylor series:

$$\begin{aligned} & V(Z(\theta), \theta) + V_\theta(Z(\theta), \theta)(\theta' - \theta) + \frac{1}{2} V_{\theta\theta}(Z(\theta), \theta)(\theta' - \theta)^2 + ... \\ \leq & V(\theta) + V_\theta(\theta)(\theta' - \theta) + \frac{1}{2} V_{\theta\theta}(\theta)(\theta' - \theta)^2 + ... \end{aligned} \tag{8.19}$$

Envelope Properties

By first-order envelope properties (8.17) and (8.18), we have

$$(V_{\theta\theta}(Z(\theta), \theta) - V_{\theta\theta}(\theta))(\theta' - \theta)^2 + ... \leq 0. \qquad (8.20)$$

Envelope Properties

- Consider θ' sufficiently close to θ, quadratic term in the expansion would dominate rest of the terms.
- For the inequality (8.20) to hold, a necessary condition is

$$V_{\theta\theta}(Z(\theta), \theta) \leq V_{\theta\theta}(\theta). \tag{8.21}$$

- This proves that long-run maximum value function is at least as convex as short-run value function at the point where the two are tangent.

Envelope Properties

For suitably “regular” maxima, we have a strict inequality in (8.21).

8.E. Examples

Example 8.1: Consumer Theory

Consider the consumer's expenditure minimization problem:

$$\min_x px \tag{EMP}$$
$$\text{s.t. } u(x) \geq u.$$

Example 8.1: Consumer Theory

- In Example 5.2, we define consumer's expenditure function $E(p, u)$ as minimum value to expenditure minimization problem (EMP) above.
- We denote optimum quantity as compensate demand function $C(p, u)$.
- Envelope property implies:

$$C(p, u) = E_p(p, u). \tag{8.22}$$

Example 8.1: Consumer Theory

- In Example 6.2, we showed that expenditure function $E(p, u)$ is concave in p.
- Now by Theorem 8.3, we know that it means that $E_{pp}(p, u)$ is negative semi-definite.
- Differentiating

$$C(p, u) = E_p(p, u) \tag{8.22}$$

with respect to p:

$$C_p(p, u) = E_{pp}(p, u). \tag{8.23}$$

Example 8.1: Consumer Theory

(i) Because second derivative matrix $E_{pp}(p, u)$ is symmetric by Schwarz's theorem, $C_p(p, u)$ is symmetric:

$$\frac{\partial C^j}{\partial p_k} = \frac{\partial C^k}{\partial p_j} = E_{jk}.$$

This is symmetry of substitution effects of price changes.

Example 8.1: Consumer Theory

(ii)
- $E_{pp}(p,u)$ is negative semi-definite.
- That is, $y^T E_{pp}(p,u) y \leq 0$ for all $y \in \mathbb{R}^n$.
- In particular, we could choose $y = e^j$
- Then $e^{j^T} E_{pp}(p,u) e^j = E_{jj} \leq 0 \implies \dfrac{\partial C^j}{\partial p_j} \leq 0. \quad (8.24)$
- This is true for all j.
- Therefore, own substitution effects of price changes are non-positive.

Example 8.1: Consumer Theory

- Second result follows even more simply from the very concept of maximum.
- For interested students, please refer to textbook or lecture notes.

Example 8.2: The LeChatelier Samuelson Principal

- Consider consumer's expenditure minimization problem (EMP) again.
- Now, we focus on second-order envelope properties.
- Consider a change in p_1 and compare the following two situations:

  (i) Quantities of all goods are free to change optimally;

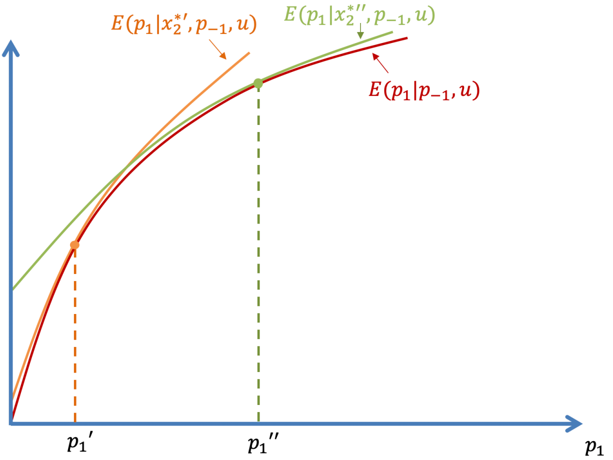
  (ii) Quantity x_2 must be kept fixed at its initially optimal level.

Example 8.2: The LeChatelier Samuelson Principal

- Let $E(p_1 \mid p_{-1}, u)$ denotes the expenditure function in situation (i) and $E(p_1 \mid x_2, p_{-1}, u)$ denotes the expenditure function in situation (ii) where x_2 must be kept fixed.

- Let $C(p_1 \mid p_{-1}, u)$ and $C(p_1 \mid x_2, p_{-1}, u)$ be the corresponding compensated demand.

Example 8.2: The LeChatelier Samuelson Principal



Example 8.2: The LeChatelier Samuelson Principal

Envelope properties of the curves:

1. First-order envelope property shows that curves will be tangential at point where x_2 is at its optimal value;
2. Second-order envelope property shows that $E(p_1 \mid p_{-1}, u)$ is more concave than $E(p_1 \mid x_2^{*\prime}, p_{-1}, u)$ and $E(p_1 \mid x_2^{*\prime\prime}, p_{-1}, u)$:

$$E_{p_1 p_1}(p_1 \mid p_{-1}, u) \leq E_{p_1 p_1}(p_1 \mid x_2^{*\prime}, p_{-1}, u)$$

$$\text{and } E_{p_1 p_1}(p_1 \mid p_{-1}, u) \leq E_{p_1 p_1}(p_1 \mid x_2^{*\prime\prime}, p_{-1}, u).$$

Example 8.2: The LeChatelier Samuelson Principal

- We know from (8.23) in Example 8.1 that

$$C^1_{p_1}(p_1 \mid p_{-1}, u) = E_{p_1 p_1}(p_1 \mid p_{-1}, u)$$

$$C^1_{p_1}(p_1 \mid x_2, p_{-1}, u) = E_{p_1 p_1}(p_1 \mid x_2, p_{-1}, u)$$

- Therefore, $C^1_{p_1}(p_1 \mid p_{-1}, u) \leq C^1_{p_1}(p_1 \mid x_2, p_{-1}, u)$

$$\underbrace{\Longrightarrow}_{C^1_{p_1}(p_1 | p_{-1}, u) \leq 0, C^1_{p_1}(p_1 | x_2, p_{-1}, u) \leq 0} \left| C^1_{p_1}(p_1 \mid p_{-1}, u) \right| \geq \left| C^1_{p_1}(p_1 \mid x_2, p_{-1}, u) \right|$$

i.e.,

$$\left| \frac{\partial x_1}{\partial p_1} \right|_{x_2 \text{ free}} \geq \left| \frac{\partial x_1}{\partial p_1} \right|_{x_2 \text{ fixed}} \qquad (8.25)$$

Example 8.2: The LeChatelier Samuelson Principal

- Fixing quantity of some other good 2 makes compensated demand for good 1 less responsive to its own price.
- Roughly speaking, any imposed rigidity in one sector of economy causes a reduction in the responsiveness to prices in other sectors.
- This is true irrespective of whether good 1 and good 2 are substitutes or complements.
- This is known as LeChatelier Samuelson Principle.

Example 8.4: Use of Second-order Conditions (Part I)

- Consider a firm that buys a vector x of inputs at prices w, produced output $y = f(x)$, and sells it for revenue $R(y)$.

- Firm's profit maximization problem is

$$\max_x F(x, w) \equiv \max_x R(f(x)) - wx,$$

where w is a row vector of input prices.

Example 8.4: Use of Second-order Conditions (Part I)

Result:

- Second-order necessary condition implies

$$\mathrm{d}w\mathrm{d}x^* = \mathrm{d}w F_{xx}(x^*, w)^{-1}\mathrm{d}w^T \leq 0.$$

- If the maximum is "regular", that is, second-order sufficient condition is satisfied, then

$$\mathrm{d}w\mathrm{d}x^* < 0.$$

Example 8.4: Use of Second-order Conditions (Part II)

- Consider the consumer's utility maximization problem:

$$\max_x U(x)$$

$$\text{s.t. } px = I.$$

- We want to find pure substitution effect of a price change.

Example 8.4: Use of Second-order Conditions (Part II)

Result:

- If the second-order sufficient condition "$\mathcal{L}_{xx}(x^*, \lambda^*)$ is negative definite" is satisfied, then

$$\mathrm{d}p\mathrm{d}x^* < 0.$$

- That is, sign of own substitution effect is negative.