Chapter 7. Concave Programming

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Introduction

- In this chapter, we will combine the idea of convexity with a more conventional calculus approach.

- The result is that the Lagrange or Kuhn-Tucker conditions, in conjunction with convexity properties of the objective and constraint functions, are sufficient for optimality.

7.A. Concave Functions and Their Derivatives

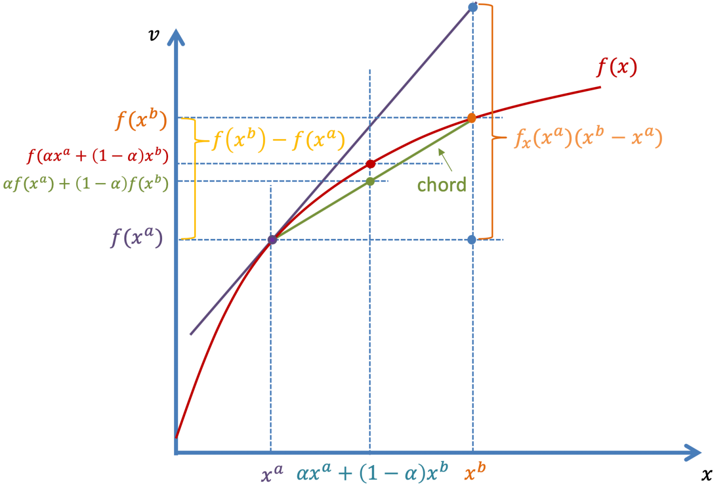
- The first step is to express the concavity (convexity) of functions in terms of their derivatives.

Definition 6.B.5 (Concave Function)**.** A function $f : \mathcal{S} \rightarrow \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, is concave if

$$f(\alpha x^a + (1-\alpha)x^b) \geq \alpha f(x^a) + (1-\alpha)f(x^b), \tag{6.5}$$

for all x^a, $x^b \in \mathcal{S}$ and for all $\alpha \in [0, 1]$.

Concave Function



Concave Function

- To express the concavity of $f(x)$ in terms of its derivative, we now draw the tangent to $f(x)$ at x^a.

- The requirement of concavity says that the graph of the function should lie on or below the tangent.

- Or expressed differently,

$$f_x(x^a)(x^b - x^a) \geq f(x^b) - f(x^a),$$

where $f_x(x^a)$ is the slope of the tangent to $f(x)$ at x^a.

Concave Function

- Such an expression holds for higher dimensions.
- The result is summarized in Proposition 7.A.1 below.

Proposition 7.A.1 (Concave Function)**.** A differentiable function $f : \mathcal{S} \to \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, is concave if and only if

$$f_x(x^a)(x^b - x^a) \geq f(x^b) - f(x^a), \tag{7.1}$$

for all $x^a, x^b \in \mathcal{S}$.

Convex Function

Similarly, for a differentiable convex function f, we have

$$f_x(x^a)(x^b - x^a) \leq f(x^b) - f(x^a). \tag{7.2}$$

7.B. Concave Programming

- A particularly important class of optimization problems has a concave objective function and convex constraint functions.

- The term concave programming is often used to describe the general problem of this kind.

Concave Programming

Consider the maximization problem

$$\max_x F(x)$$
$$\text{s.t. } G(x) \le c,$$

where F is differentiable and concave, and each component constraint function G^i is differentiable and convex.

Concave Programming

We will interpret the problem using the terminology of the production problem:

$$\max_x \underbrace{F(x)}_{\text{revenue from outputs}}$$

$$\text{s.t. } \underbrace{G(x) \leq c}_{\text{resource constraints}},$$

- x: the vector of outputs
- c: a fixed vector of input supplies
- $G(x)$: the vector of inputs needed to produce x
- $X(c)$: the optimum choice funcion
- $V(c)$: the maximum value function

Claim 1. $V(c)$ is a non-decreasing function.

- feasible x for a given c remains feasible when any component of c increases, so maximum value cannot decrease.

Concave Programming

Claim 2. $V(c)$ is a concave.

To show concavity of $V(c)$, we need to show:

for any two input supply vectors c and c' and any number $\alpha \in [0, 1]$, we have

$$V(\alpha c + (1 - \alpha)c') \geq \alpha V(c) + (1 - \alpha)V(c').$$

Claim 2: $V(c)$ is a concave (Intuition)

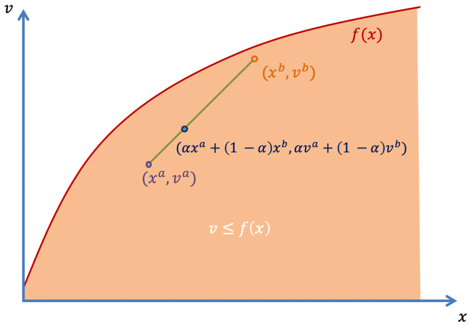
- Convexity of G rules out economies of scale or specialization in production, ensuring that a weighted average of outputs can be produced using the same weighted average of inputs.

- Concavity of F ensures that the resulting revenue is at least as high as the same weighted average of the separate revenues.

Concave Function

Recall the alternative interpretation of a concave function:

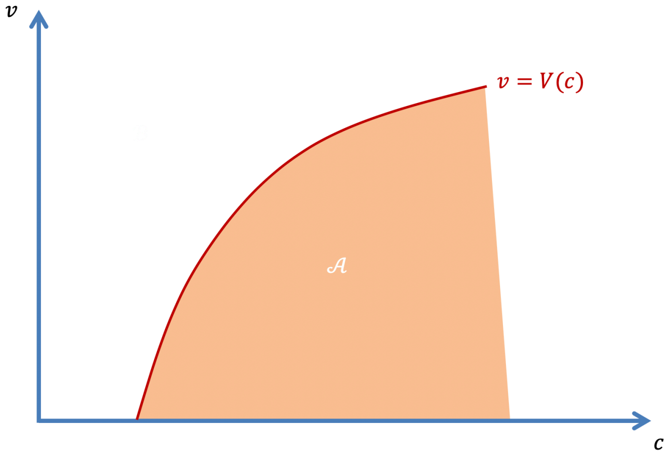
Claim. f is a concave function if and only if $\mathcal{F} = \{(x, v) | v \leq f(x)\}$ is a convex set.



Concave Function

- In our current context, as $V(c)$ is a concave function, the set $\{(c, v)|v \leq V(c)\}$ is a convex set.
- This is an $(m + 1)$-dimensional set, the collection of all points (c, v) such that $v \leq V(c)$.
- That is, revenue of v can be produced using the input vector c.

Non-decreasing and Concave $V(c)$



Separation

- Since $\mathcal{A}$ is a convex set, it can be separated from other convex sets.
- Choose a point $(c^*, v^*) \in \mathcal{A}$ such that $v^* = V(c^*)$.
- (c^*, v^*) must be a boundary point since for any $r > 0$, there exists $\varepsilon \in (0, r)$

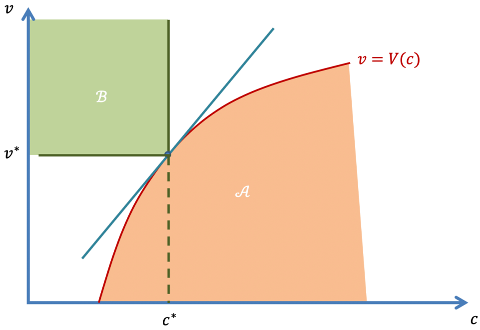
  (i) $v^* - (r - \varepsilon) < v^* = V(c^*)$ implies that the point $(c^*, v^* - (r - \varepsilon))$ is in $\mathcal{A}$;

  (ii) $v^* + (r - \varepsilon) > v^* = V(c^*)$ implies that the point $(c^*, v^* + (r - \varepsilon))$ is not in $\mathcal{A}$.

Separation

- Define $\mathcal{B}$ as the set of all points (c, v) such that

$$c \leq c^* \text{ and } v \geq v^*.$$



Separation

- $\mathcal{B}$ is a convex set.

- $\mathcal{A}$ and $\mathcal{B}$ have no common interior points.

Separation

- We could apply Separation Theorem.
- (c^*, v^*) is a common boundary point of $\mathcal{A}$ and $\mathcal{B}$.
- We could write the equation of the separating hyperplane as follows: $\iota v - \lambda c = b = \iota v^* - \lambda c^*$, where ι is a scalar, and λ is a m-dimensional row vector.
- The signs are so chosen that

$$\iota v - \lambda c \begin{cases} \leq b & \text{for all } (c, v) \in \mathcal{A} \\ \geq b & \text{for all } (c, v) \in \mathcal{B}. \end{cases} \tag{7.6}$$

Separation

Remark. ι and λ must both be non-negative.

Separation

Now comes the more subtle question:

Question. Can ι be zero?

Consequence of $\iota = 0$

(i) • For $\iota v - \lambda c = b$ to be meaningful, (ι, λ) must be non-zero.

• Therefore, $\lambda_i \neq 0$ for at least one i.

• Given that $\lambda_i \geq 0$ for all i, $\lambda_i > 0$ for at least one i.

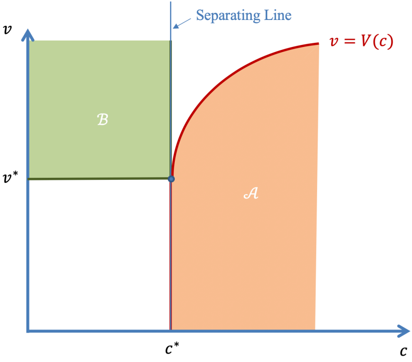
(ii) • Equation of hyperplane becomes $-\lambda c = b = -\lambda c^*$.

• For all $(c, v) \in \mathcal{A}$, $-\lambda c \leq -\lambda c^*$, or $\lambda(c - c^*) \geq 0$.

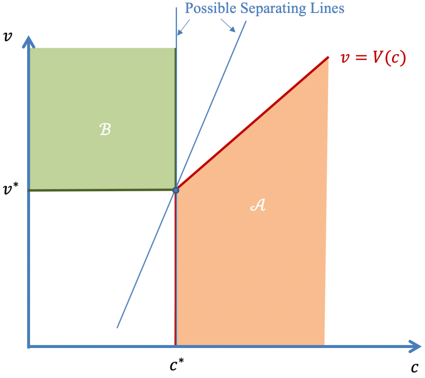
Consequence of $\iota = 0$

- In scalar constraint case, $\lambda > 0$.
- $\lambda(c - c^*) \geq 0$ implies $c - c^* \geq 0$.
- Graphically, separating line is vertical at c^*, and set $\mathcal{A}$ lies entirely to the right of it.
  - No feasible points to the left of c^*: production is impossible if input supply falls short of this level.
  - In some applications, this can happen because of indivisibilities.

Consequence of $\iota = 0$



(a)



(b)

Consequence of $\iota = 0$

As c approaches c^*,

(i) In case 7.1a, marginal revenue product goes to infinity.

- only a vertical separating line

(ii) In case 7.1b, marginal revenue product is finite.

- a vertical separating line
- many non-vertical separating lines with positive ι

Constraint Qualification

- We would like to ensure a positive ι so that marginal revenue product of a resource is finite.
- We do this by ensuring the existence of c such that $c < c^*$.
- Due to the existence of case (ii) above, such conditions are only sufficient but not necessary.

Constraint Qualification

Claim. If there exists an x^o such that $G(x^o) \ll c^*$ and $F(x^o)$ is defined, then $\iota > 0$.

- This requirement is constraint qualification for concave programming problem.
- It is sometimes called Slater condition.

Constraint Qualification: Intuition

- For scalar c, such a condition works since

  (i) $(G(x^o), F(x^o)) \in \mathcal{A}$ and

  (ii) $(G(x^o), F(x^o))$ is a point to the left of c^*.

  $(G(x^o) < c^* \ )$

- Separating line cannot have an infinite slope at c^*.

Constraint Qualification

We prove that Slater condition implies $\iota > 0$ in general.

Normalization

- Separation property (7.6) is unaffected if we multiply by b, ι and λ_i by the same positive number.
- Once we can be sure that $\iota \neq 0$, we can choose a scale to make $\iota = 1$.
- In economic terms, ι and λ constitute a system of shadow prices, ι for revenue and λ for the inputs.
- Only relative prices matter for economic decisions, in setting $\iota = 1$, we are choosing revenue to be the numéraire.
- We will adopt this normalization henceforth.

Shadow Price Interpretation of λ

- Observe that by the separation property (7.6), for all $(c, v) \in \mathcal{A}$,
$$v - \lambda c \leq v^* - \lambda c^*.$$
- That is, (c^*, v^*) achieves the maximum value of $(v - \lambda c)$ among all points $(c, v) \in \mathcal{A}$.
- If we interpret λ as the vector of shadow prices of inputs, then $(v - \lambda c)$ is the profit that accrues when a producer uses inputs c to produce revenue v.

Shadow Price Interpretation of λ

- Since all points in $\mathcal{A}$ represents feasible production plans, a profit-maximizing producer will pick (c^*, v^*).
- This means that the producer need not be aware that in fact the availability of inputs is limited to c^*.
- He may think that he is free to choose any c but ends up choosing the right c^*.
- It is the prices λ that brings home to him the scarcity.

Shadow Price Interpretation of λ

- The principle behind this interpretation is general and important: constrained choice can be converted into unconstraint choice if proper scarcity costs or shadow values of constraints are netted out of criterion function.

- As it will become clear later, this is the most important feature of Lagrange's Method in concave programming.

Generalized Marginal Products

- For any c, the point $(c, V(c))$ is in $\mathcal{A}$.
- So by the separation property, we have

$$V(c) - \lambda c \leq V(c^*) - \lambda c^*,$$

$$\text{or} \quad V(c) - V(c^*) \leq \lambda(c - c^*). \tag{7.9}$$

- If $V(c)$ is differentiable, then by Proposition 7.A.1, concavity of $V(c)$ means

$$V(c) - V(c^*) \leq V_c(c^*)(c - c^*). \tag{7.10}$$

- (7.9) and (7.10) suggest $\lambda = V_c(c^*)$ (shadow prices)

Generalized Marginal Products

- However, the problem is that V may not be differentiable.
- Let us consider a general point $(c, V(c))$ with its associated multiplier vector λ.
- Compare this with a neighboring point where only the i^{th} input is increase: $(c + he^i, V(c + he^i))$, where h is a positive scalar.

Generalized Marginal Products

- Then by separation property

$$V(c) - V(c^*) \leq \lambda(c - c^*). \tag{7.9}$$

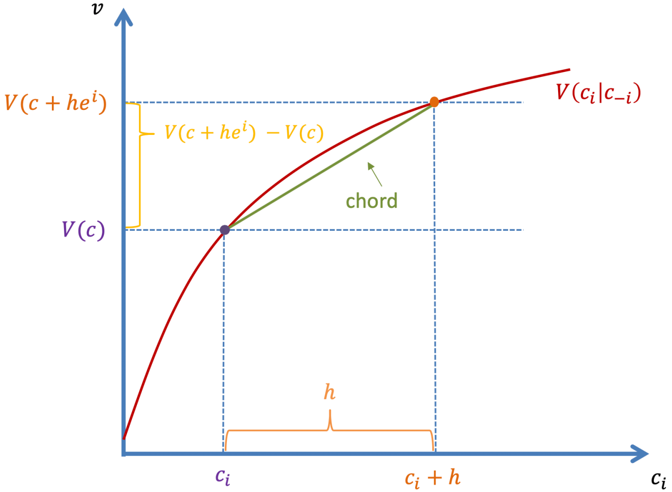
we have

$$\frac{[V(c + he^i) - V(c)]}{h} \leq \lambda_i. \tag{7.11}$$

- We show that by concavity of V, LHS of (7.11) is a non-increasing function of h.

Generalized Marginal Products

Graphically, $\frac{[V(c+he^i)-V(c)]}{h}$ is simply the slope of the chord.



Generalized Marginal Products

- Therefore, LHS expression must attain the maximum as h goes to zero from positive values.
- This limit is defined as the "rightward" partial derivative of V with respect to the i^{th} coordinate of c: $V_i^+(c)$.
- Therefore,

$$\frac{[V(c+he^i) - V(c)]}{h} \leq \lambda_i. \tag{7.11}$$

implies $V_i^+(c) \leq \lambda_i$.

Generalized Marginal Products

- Similarly, we could repeat the analysis for $h < 0$.
- Now we have

$$\frac{[V(c + he^i)) - V(c)]}{h} \geq \lambda_i. \tag{7.13}$$

- Taking the limit from the negative values of h gives the "leftward" partial derivative $V_i^-(c)$.
- This proves $V_i^-(c) \geq \lambda_i$.

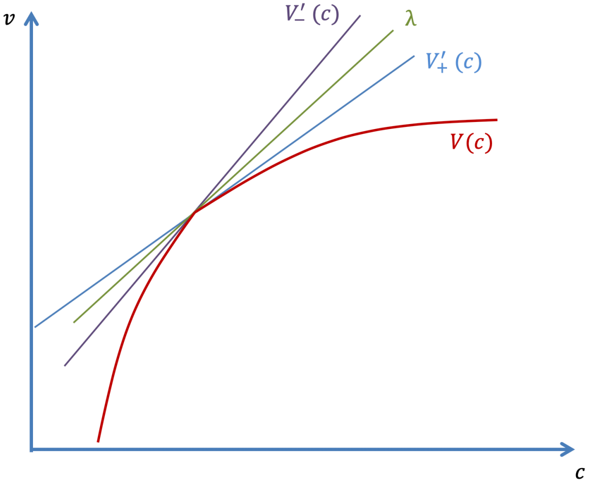
Generalized Marginal Products

- Combining the two, we have

$$V_i^-(c) \geq \lambda_i \geq V_i^+(c). \tag{7.14}$$

- This result generalizes the notion of diminishing marginal returns and relates the multipliers to these generalized marginal products.

Generalized Marginal Products



Choice Variables

- So far the vector of choice variables x has been kept in the background.

- Let's now consider it explicitly.

Choice Variables

- $(G(x^*), F(x^*)) \in \mathcal{A}$, separation property gives

$$F(x^*) - \lambda G(x^*) \leq V(c) - \lambda c \underbrace{\Longrightarrow}_{F(x^*)=V(c)} \lambda \left[c - G(x^*)\right] \leq 0$$

$$\Longrightarrow \sum_{i=1}^{m} \lambda_i \left[c_i - G^i(x^*)\right] \leq 0.$$

- Since $\lambda_i \geq 0$ and $G^i(x) \leq c_i$ for all i, we have

  $\lambda_i \left[c_i - G^i(x^*)\right] \geq 0$ for all i.

- Therefore,

$$\lambda_i \left[c_i - G^i(x^*)\right] = 0. \tag{7.15}$$

- This is just complementary slackness.

Choice Variables

- For any x, the point $(G(x), F(x)) \in \mathcal{A}$.
- Separation property gives

$$F(x) - \lambda G(x) \underbrace{\leq}_{\text{separation property}} V(c) - \lambda c \underbrace{=}_{F(x^*)=V(c) \text{ and } (7.15)} F(x^*) - \lambda G(x^*) \text{ for all } x.$$

- x^* maximizes $F(x) - \lambda G(x)$ without any constraints.
- This means that the shadow prices allow us to convert the original constrained revenue-maximization problem into an unconstrained profit-maximization problem.

Necessary Conditions for Concave Programming

Theorem 7.1 (Necessary Conditions for Concave Programming). Suppose that F is a concave function and G is a vector convex function, and that there exists an x^o satisfying $G(x^o) \ll c$. If x^* maximizes $F(x)$ subject to $G(x) \leq c$, then there is a row vector λ such that

(i) x^* maximizes $F(x) - \lambda G(x)$ without any constraints, and

(ii) $\lambda \geq 0$, $G(x^*) \leq c$ with complementary slackness.

Necessary Conditions for Concave Programming

- Theorem 7.1 does not require F and G to have derivatives.
- But if the functions are differentiable, then we have first-order necessary conditions for maximization problem (i):

$$F_x(x^*) - \lambda G_x(x^*) = 0. \tag{7.16}$$

- In terms of the Lagrangian $\mathcal{L}(x, \lambda)$, (7.16) becomes $\mathcal{L}_x(x^*, \lambda)$.
- This is just condition of Lagrange's Theorem.
- We could further add non-negativity constraints on x, and get Kuhn-Tucker Theorem.

Necessary Conditions for Concave Programming

- Concave programming goes beyond general Lagrange or Kuhn-Tucker conditions.
- In general, there was no claim that x^* maximized the Lagrangian.
- However, when F is concave and G is convex, part (i) of Theorem 7.1 is easily transformed into $\mathcal{L}(x, \lambda) \leq \mathcal{L}(x^*, \lambda)$ for all x, so x^* does maximize the Lagrangian.

Necessary Conditions for Concave Programming

Our interpretation of Lagrange's method as converting the constrained revenue-maximization into unconstrained profit-maximization must be confined to the case of concave programming.

Sufficient Conditions for Concave Programming

- First-order necessary conditions are sufficient to yield a true maximum in the concave programming problem.
- The argument proceeds in two parts.

  1. Suppose x^* satisfies (i) and (ii) in Theorem 7.1, then x^* maximizes $F(x)$ subject to $G(x) \leq c$.

  2. Suppose x^* satisfies first-order condition (and F concave, G convex), then (i) holds.

Sufficient Conditions for Concave Programming

Theorem 7.2 (Sufficient Conditions for Concave Programming). If x^* and λ are such that

(i) x^* maximizes $F(x) - \lambda G(x)$ without any constraints, and

(ii) $\lambda \geq 0$, $G(x^*) \leq c$ with complementary slackness,

then x^* maximizes $F(x)$ subject to $G(x) \leq c$. If $F - \lambda G$ is concave (for which in turn it suffices to have F concave and G convex), then

$$F_x(x^*) - \lambda G_x(x^*) = 0 \tag{7.16}$$

implies (i) above.

Sufficient Conditions for Concave Programming

Note that no constraint qualification appears in the sufficient conditions.

7.C. Quasi-concave Programming

- In the separation approach of Chapter 6, F was merely quasi-concave and each component constraint function in G was quasi-convex.
- In this chapter, the stronger assumption of concavity and convexity has been made so far.

Quasi-concave Programming

- In fact, the weaker assumptions of quasi-concavity (quasi-convexity) make little difference to necessary conditions.
- They yield sufficient conditions like the ones above for concave programming, but only in the presence of some further technical conditions that are complex to establish.
- For interested students, please refer to the paper "Arrow and Enthoven (1961). Quasi-concave Programming. Econometrica, 779-800."

Quasi-concave Programming

We will discuss only a limited version of quasi-concave programming, namely, the one where objective function is quasi-concave and constraint function is linear:[1]

$$\max_x F(x) \tag{MP1}$$

$$\text{s.t. } px \leq b,$$

where p is a row vector and b is a number.

[1] The mirror-image case of a linear objective and a quasi-convex constraint can be treated in the same way.

Quasi-concave Programming

Recall the definition of Quasiconcavity:

Definition 6.B.3 (Quasi-concave Function)**.** A function $f : \mathcal{S} \to \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, quasi-concave

- if the set $\{x | f(x) \geq c\}$ is convex for all $c \in \mathbb{R}$,
- or equivalently, if $f(\alpha x^a + (1-\alpha)x^b) \geq \min\{f(x^a), f(x^b)\}$, for all x^a, x^b and for all $\alpha \in [0, 1]$.

Quasi-concave Programming

We need to establish some property of quasi-concave function, relating to the derivatives.

For a quasi-concave differentiable function $F : \mathcal{S} \to \mathbb{R}$,

$$F_x(x^a)(x^b - x^a) \geq 0. \tag{7.21}$$

for all x^a, x^b such that $F(x^b) \geq F(x^a)$.

Quasi-concave Programming

- Now consider the maximization problem

$$\max_x F(x) \tag{MP1}$$

$$\text{s.t. } px \leq b,$$

- First-order necessary conditions are

$$F_x(x^*) - \lambda p = 0 \tag{7.22}$$

$$px^* \leq b \text{ and } \lambda \geq 0, \text{ with complementary slackness}$$

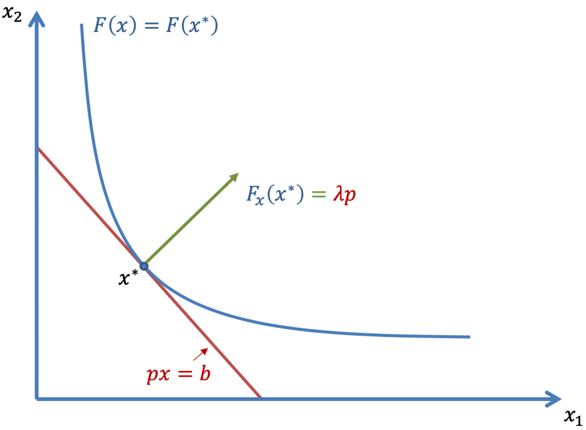
Quasi-concave Programming

We claim that (7.22) is also sufficient when $\lambda > 0$ and the constraint is binding.[2] Formally,

Claim. If F is continuous and quasi-concave, x^* and $\lambda > 0$ satisfy first-order necessary conditions, then x^* solves the quasi-concave programming problem.

[2] Appendix B provides an example of a spurious stationary point where (7.22) holds with $\lambda = 0$.

Quasi-concave Programming



Quasi-concave Programming

- $F_x(x^*)$ is normal to the contour of $F(x)$ at x^*.
- p is normal to the constraint $px = b$ at x^*.
- The usual tangency condition is equivalent to the normal vectors being parallel.
- Equation (7.22) expresses this, with the constant of proportionality equal to λ.

7.D. Uniqueness

- The above sufficient conditions for concave as well as quasi-concave programming are weak in the sense that they establish that no other feasible choice x can do better than x^*.

- They do not rule out existence of other feasible choices that yield $F(x) = F(x^*)$.

- In other words, they do not establish the uniqueness of the optimum.

Uniqueness

As discussed in Chapter 6, a strenghening of the concept of concavity or quasi-concavity gives uniqueness.

Definition 7.D.1 (Strictly Concave Function)**.** A function $f : \mathcal{S} \to \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, is strictly concave if

$$f(\alpha x^a + (1-\alpha)x^b) > \alpha f(x^a) + (1-\alpha) f(x^b), \qquad (7.24)$$

for all x^a, $x^b \in \mathcal{S}$ and for all $\alpha \in (0, 1)$.

Uniqueness

Claim. If objective function F in concave programming problem is strictly concave, then maximizer x^* is unique.

Proof by contradiction.

7.E. Examples

Example 7.1: Linear Programming

An important special case of concave programming is the theory of linear programming.

$$\max_{x} F(x) \equiv ax \qquad \text{(Primal)}$$

$$\text{s.t. } G(x) \equiv Bx \leq c \text{ and } x \geq 0,$$

where a is an n-dimensional row vector and B an m-by-n matrix.

Example 7.1: Linear Programming

- Now

$$F_x(x) = a \text{ and } G_x(x) = B.$$

- When the constraint functions are linear, no constraint qualification is needed.

- All conditions of concave programming are fulfilled, and Kuhn-Tucker conditions are both necessary and sufficient.

Example 7.1: Linear Programming

- The Lagrangian is

$$\mathcal{L}(x, \lambda) = ax + \lambda[c - Bx]. \tag{7.25}$$

- The optimum x^* and λ^* satisfy Kuhn-Tucker conditions:

$$a - \lambda^* B \leq 0,\ x^* \geq 0, \text{with complementary slackness}, \tag{7.26}$$

$$c - Bx^* \geq 0,\ \lambda^* \geq 0, \text{with complementary slackness}. \tag{7.27}$$

Example 7.1: Linear Programming

- (7.26) and (7.27) contain 2^{m+n} combinations of patterns of equations and inequalities.

- As a special feature of the linear programming problem, if k of the constraints in (7.27) hold with equality, then exactly $(n-k)$ non-negativity constraints in (7.26) should bind.

- When this is the case, the corresponding equations for λ is also of the correct number m.

Example 7.1: Linear Programming

Next, consider a new linear programming problem:

$$\max_{y} -yc \tag{Dual}$$

$$\text{s.t.} \quad -yB \leq -a \text{ and } y \geq 0,$$

where y is a m-dimensional row vector and vectors a, c and matrix B are exactly as before.

Example 7.1: Linear Programming

- We introduce a column vector μ of multipliers and define the Lagrangian:

$$\mathcal{L}(x, \lambda) = -yc + [-a + yB]\mu. \tag{7.28}$$

- Optimum y^* and μ^* satisfy the necessary and sufficient Kuhn-Tucker conditions:

$$-c + B\mu^* \leq 0,\ y^* \geq 0, \text{with complementary slackness,} \tag{7.29}$$

$$-a + y^*B \geq 0,\ \mu^* \geq 0, \text{with complementary slackness.} \tag{7.30}$$

Example 7.1: Linear Programming

- (7.29) is exactly the same as (7.27) and (7.30) is exactly the same as (7.26), if we replace y^* by λ^* and μ^* by x^*.
- In other words, optimum x^* and λ^* solve new problem.
- New problem is said to be dual to the original, which is then called the primal problem in the pair.
- This captures an important economic relationship between prices and quantities in economics.

Example 7.1: Linear Programming

- We interpret the primal problem as follows:

$$\max_x \underbrace{a}_{\text{output prices}} \underbrace{x}_{\text{output quantities}}$$

$$\text{s.t. } \underbrace{Bx}_{\text{inputs for producing } x} \leq \underbrace{c}_{\text{input supplies}} \text{ and } x \geq 0,$$

- Solving the primal problem, we get x^* and λ^*.

- λ^* is vector of shadow prices of the inputs.

Example 7.1: Linear Programming

- Rewriting dual problem in terms of λ.

- From previous analysis, λ^* solves dual problem.

$$\lambda^* = \min_{\lambda}\{\lambda c \mid \lambda B \geq a \text{ and } \lambda \geq 0\}$$

- Thus, shadow prices minimize cost of the input c.

Example 7.1: Linear Programming

- j^{th} component of λB is $\sum_i \lambda_i B_{ij}$: cost of bundle of inputs needed to produce one unit of good j, calculated using shadow prices.
- Constraint $\sum_i \lambda_i B_{ij} \geq a_j$: input cost of good j is at least as great as unit value of output of good j. This is true for all good j.
- In other words, shadow prices of inputs ensure that no good can make a strictly positive profit – a standard "competitive" condition in economics.

Example 7.1: Linear Programming

Complementary slackness in (7.26) ensures that

(i) If unit cost of production of j, $\sum_i \lambda_i B_{ij}$, exceeds its prices a_j, then $x_j = 0$. That is, if production of j would entail a loss when calculated using the shadow prices, then good j would not be produced.

(ii) If good j is produced in positive quantity, $x_j > 0$, then unit cost exactly equals the price, $\sum_i \lambda_i B_{ij} = a_j$. That is, profit is exactly 0.

Example 7.1: Linear Programming

- Complementary slackness in (7.26) and (7.27) imply

$$[a - \lambda^* B]x^* = 0 \implies ax^* = \lambda^* Bx^*$$

$$\lambda^*[c - Bx^*] = 0 \implies \lambda^* c = \lambda^* Bx^*$$

- Combining the two, we have $ax^* = \lambda^* c$ (7.31)
- This says that value of optimum output equals cost of factor supplies.
- This result can be interpreted as circular flow of income, that is, national product equals national income.

Example 7.1: Linear Programming

- Finally, it is easy to check that if we take dual problem as our starting-point and go through mechanical steps to finding its dual, we return to primal.
- In other words, duality is reflexive.

Example 7.1: Linear Programming

- This is the essence of the duality theory of linear programming.
- One final remark is that we took optimum x^* as our starting point, however, solution may not exist, because constraints may be mutually inconsistent, or they may define an unbounded feasible set.
- This issue beyond our discussion here and is left to more advanced texts.

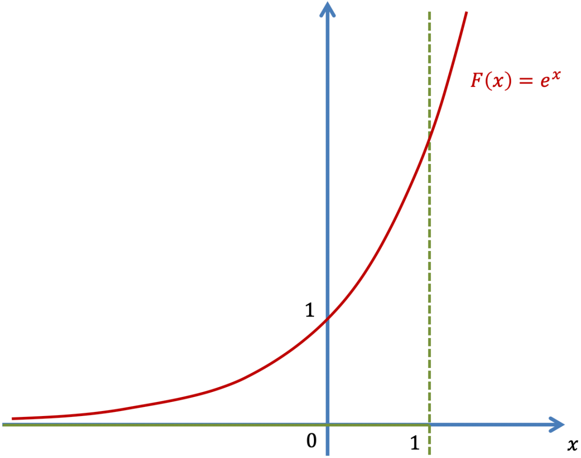
Example 7.2: Failure of Profit-maximizing

For a scalar x, consider the following maximization problem:

$$\max_x F(x) \equiv e^x$$
$$\text{s. t. } G(x) \equiv x \leq 1.$$

$F(x)$ is increasing, and maximum occurs at $x = 1$.

Example 7.2: Failure of Profit-maximizing



Example 7.2: Failure of Profit-maximizing

- Kuhn-Tucker Theorem applies.
- Lagrangian is

$$\mathcal{L}(x, \lambda) = e^x + \lambda(1 - x).$$

- Kuhn-Tucker necessary conditions are

$\partial\mathcal{L}/\partial x = e^x - \lambda = 0;$

$\partial\mathcal{L}/\partial\lambda = 1 - x \geq 0$ and $\lambda \geq 0,$ with complementary slackness.

- Solution is $x^* = 1$ and $\lambda = e$.

Example 7.2: Failure of Profit-maximizing

- However, $x = 1$ does not maximize $F(x) - \lambda G(x)$ without constraints.

- In fact, $e^x - ex$ can be made arbitrarily large by increasing x beyond 1.

- Here, Lagrange's method does not convert original constrained maximization problem into an unconstrained profit-maximization problem, because F is not concave.