# Chapter 11. Dynamic Programming

The main reference of this chapter is

- Chapters 2 to 5 of Stokey, N. L., Lucas, R. E., & Prescott, E. C. (1989). *Recursive Methods in Economic Dynamics*. Harvard University Press.

We will not provide rigorous mathematical proofs. For interested students, please refer to chapters 3 and 4 of Stokey, Lucas & Prescott (1989).

## 11.A. Life-cycle saving problem revisited

We consider an extremely simplified version of the life-cycle saving problem introduced in Section 10.A of Chapter 10. In particular, assume

1. wage $w_t$ is 0, i.e., $w_t = 0$ for all $t$;
2. interest rate is 0, i.e., $r_t = 0$ for all $t$;
3. utility function takes the form $u(c) = \ln(c)$;
4. no discouting, i.e., $\beta = 1$;
5. terminal stock $k_{T+1} = 0$.

The problem is restated below with the above assumptions imposed.

**Life-cycle saving problem (finite-horizon)** Time is discrete and denoted by $t = 0, 1, 2, ..., T$. The decision is on how much of the income to spend on consumption in each period. The unspent income is saved and the overspent income is on debt. Let $c_t \geq 0$ be the consumption in period $t$ and $k_{t+1}$ be the accumulated savings or debts at the beginning of period $t+1$. The budget constraint in period $t$ is

$$c_t + k_{t+1} = k_t. \tag{11.1}$$

$k_0 > 0$ is given. Furthermore, $k_{T+1} = 0$ is imposed.

The individual only derives utility from consumption and chooses the consumption path to maximize the total value of utilities in period $t = 0$:

$$U(c_0, c_1, ..., c_T) = \sum_{t=0}^{T} \ln(c_t).$$

**The maximization principle** The maximization problem is:

$$\max_{\substack{c_0,c_1,...,c_T \\ k_1,k_2,...,k_T}} \sum_{t=0}^{T} \ln(c_t) \tag{11.2}$$

$$\text{s.t. } c_t + k_{t+1} = k_t \text{ for all } t = 0, ..., T$$

We could solve the problem using the method learned in Chapter 10.

Define Hamiltonian:

$$H(c_t, k_t, \pi_{t+1}, t) = \ln(c_t) + \pi_{t+1}(-c_t)$$

FOCs are:

$$\frac{\partial H}{\partial c_t} = \frac{1}{c_t} - \pi_{t+1} = 0 \text{ for all } t = 0, ..., T$$
$$\pi_{t+1} - \pi_t = -\frac{\partial H^*}{\partial k_t} = 0 \text{ for all } t = 1, ..., T$$
$$k_{t+1} - k_t = \frac{\partial H^*}{\partial \pi_{t+1}} = -c_t \text{ for all } t = 0, ..., T \text{ (inter-temporal constraints)}$$

1. **Euler Equation**:

$$c_{t+1} = c_t \text{ for all } t = 0, ..., T \implies c_t = c_0 \text{ for all } t = 0, ..., T$$

2. From the constraints:

$$c_0 + k_1 = k_0$$
$$c_1 + k_2 = k_1$$
$$\cdots$$
$$c_T + 0 = k_T$$

   Summing up, we have $\sum_{t=0}^{T} c_t = k_0$.

3. From 1 and 2, the solution is $c_t^* = \frac{k_0}{T+1}$ for all $t = 0, ..., T$ and $k_{t+1}^* = \frac{T-t}{T+1} k_0$ for all $t = 0, ..., T-1$.

**Define the problem recursively** Now let us look at the problem from a different angle. Define the maximum value at $t = 0$ as a function of the initial stocks:

$$V_0(k_0) = \max_{\substack{c_0,c_1,...,c_T \\ k_1,k_2,...,k_T}} \{u(c_0) + u(c_1) + ... + u(c_T)\}$$

subject to budget constraints (11.1) for all $t = 0, ..., T$ and terminal condition $k_{T+1} = 0$.

Then by the previously calculated optimal consumption path $c_t^* = \frac{k_0}{T+1}$ for all $t = 0, ..., T$, we have

$$V_0(k_0) = (T+1)\ln\Big(\frac{k_0}{T+1}\Big).$$

Given $k_1$, we could similarly define the maximum value at $t = 1$ as a function of $k_1$:

$$V_1(k_1) = \max_{\substack{c_1,...,c_T \\ k_2,...,k_T}} \{u(c_1) + u(c_2) + ... + u(c_T)\}$$

subject to the budget constraints (11.1) for all $t = 1, ..., T$ and the terminal condition $k_{T+1} = 0$. Then, we could use the maximum principle to solve this new problem. This new problem only differs from the previous problem in that there is one less period and the initial stock is $k_1$ instead of $k_0$. The maximum value is

$$V_1(k_1) = T\ln\Big(\frac{k_1}{T}\Big).$$

Next, consider a two-period problem:

$$W(k_0) = \max_{c_0,k_1}\{\ln(c_0) + V_1(k_1)\big)\} = \max_{c_0,k_1}\{\ln(c_0) + T\ln\Big(\frac{k_1}{T}\Big)\}$$

$$\text{s.t. } c_0 + k_1 = k_0$$

To solve the problem, we could substitute the constraint into the objective function:

$$W(k_0) = \max_{k_1}\{\ln(k_0 - k_1) + T\ln\Big(\frac{k_1}{T}\Big)\}$$

FOC gives

$$-\frac{1}{k_0 - k_1} + \frac{T}{k_1} = 0 \implies k_1 = \frac{T}{T+1}k_0.$$

Plugging into the value function, we have

$$W(k_0) = (T+1)\ln\Big(\frac{k_0}{T+1}\Big) = V_0(k_0).$$

It suggests:

$$V_0(k_0) = \max_{c_0,k_1}\{\ln(c_0) + V_1(k_1)\}$$

$$\text{s.t. } c_0 + k_1 = k_0$$

Similarly, we could define

$$V_2(k_2) = \max_{\substack{c_2,\ldots,c_T \\ k_3,\ldots,k_T}} \{u(c_2) + u(c_3) + \ldots + u(c_T)\}$$

subject to the budget constraints (11.1) for all $t = 2, \ldots, T$ and the terminal condition $k_{T+1} = 0$, and verify

$$V_1(k_1) = \max_{c_1, k_2} \{\ln(c_1) + V_2(k_2)\}$$
$$\text{s.t. } c_1 + k_2 = k_1$$

This argument works for all $t = 0, \ldots, T-1$:

$$V_t(k_t) = \max_{c_t, k_{t+1}} \{\ln(c_t) + V_{t+1}(k_{t+1})\} \tag{11.3}$$
$$\text{s.t. } c_t + k_{t+1} = k_t$$

Therefore, for this simple problem, the equation (11.3) holds. This equation, called **Bellman Equation**, expresses the value function as a combination of a flow payoff and a (discounted) continuation payoff. Such a method of optimization over time as a succession of static programming problems is called **Dynamic Programming**.

**Life-cycle saving problem (infinite-horizon)** Bellman Equation holds for infinite-horizon problems as well. As an example, we consider an infinite-horizon version of this simplified life-cycle saving problem. For the problem to be well-defined, we need discounting. Let the discount factor be $\beta \in (0, 1)$. So the objective function becomes

$$U(c) = \sum_{t=0}^{\infty} \beta^t \ln(c_t).$$

The budget constraint in period $t$ is still

$$c_t + k_{t+1} = k_t. \tag{11.4}$$

$k_0 > 0$ is given.

**The maximization principle** Define Hamiltonian:

$$H(c_t, k_t, \pi_{t+1}, t) = \beta^t \ln(c_t) + \pi_{t+1}(-c_t)$$

FOCs are:

$$\frac{\partial H}{\partial c_t} = \beta^t \frac{1}{c_t} - \pi_{t+1} = 0 \text{ for all } t = 0, ..., T$$
$$\pi_{t+1} - \pi_t = -\frac{\partial H^*}{\partial k_t} = 0 \text{ for all } t = 1, ..., T$$
$$k_{t+1} - k_t = \frac{\partial H^*}{\partial \pi_{t+1}} = -c_t \text{ for all } t = 0, ..., T \text{ (inter-temporal constraints)}$$

We also need the transversality condition $\lim_{T\to\infty} \pi_{T+1} k_{t+1} = 0$.

1. **Euler Equation**:

$$c_{t+1} = \beta c_t \text{ for all } t = 0, ..., T \implies c_t = \beta^t c_0 \text{ for all } t = 0, ..., T$$

2. From the constraints: $\sum_{t=0}^{\infty} c_t + \lim_{T\to\infty} k_{T+1} = k_0$.
3. $\pi_{t+1} = \beta^t / c_t$ and transversality condition $\implies \lim_{T\to\infty} \frac{\beta^T k_{T+1}}{c_T} = 0$. By 1, $c_T = \beta^T c_0$. So, we have $\lim_{T\to\infty} \frac{k_{T+1}}{c_0} = 0 \implies \lim_{T\to\infty} k_{T+1} = 0$.
4. From 1, 2 and 3,

$$\sum_{t=0}^{\infty} \beta^t c_0 = k_0 \implies c_0 = (1-\beta)k_0.$$

   We further have $c_t = \beta^t c_0 = \beta^t(1-\beta)k_0$ and $k_{t+1} = k_0 - \sum_{s=0}^{t} c_s = \beta^{t+1} k_0$. Thus, in each period $t$,

$$c_t = (1-\beta)k_t \text{ and } k_{t+1} = \beta k_t.$$

   The above two equations that express $c_t$ and $k_{t+1}$ as functions of $k_t$ are called **policy functions**.

**Define the problem recursively** Similar to the finite-horizon case, we show that the Bellman Equation holds:

$$V_t(k_t) = \max_{c_t, k_{t+1}} \{\ln(c_t) + \beta V_{t+1}(k_{t+1})\} \tag{11.5}$$
$$\text{s.t. } c_t + k_{t+1} = k_t$$

In period $t$, the value function is

$$V_t(k_t) = \max_{\substack{\{c_{t+j}\}_{j=0}^{\infty} \\ \{k_{t+j+1}\}_{j=0}^{\infty}}} \sum_{j=0}^{\infty} \beta^j \ln(c_{t+j})$$

subject to budget constraints

$$c_{t+j} + k_{t+j+1} = k_{t+j}$$

for all $j \geq 0$. We could drop the time subscript $t$ in $V_t$ since functional forms of the value functions are the same in each period.

Solving the problem using the maximum principle, we have

$$c_{t+j} = \beta^j (1-\beta) k_t,$$

which gives the value function

$$V(k_t) = \frac{\ln(1-\beta) + \ln(k_t)}{1-\beta} + \frac{\beta \ln(\beta)}{(1-\beta)^2}. \tag{11.6}$$

Now define

$$W(k_t) = \max_{c_t, k_{t+1}} \{\ln(c_t) + \beta V(k_{t+1})\}$$
$$\text{s.t. } c_t + k_{t+1} = k_t$$

To solve the problem, we could substitute the constraint into the objective function:

$$W(k_t) = \max_{k_{t+1}} \Big\{ \ln(k_t - k_{t+1}) + \beta \Big[ \frac{\ln(1-\beta) + \ln(k_{t+1})}{1-\beta} + \frac{\beta \ln(\beta)}{(1-\beta)^2} \Big] \Big\}$$

FOC gives

$$-\frac{1}{k_t - k_{t+1}} + \frac{\beta}{(1-\beta) k_{t+1}} = 0 \implies k_{t+1} = \beta k_t.$$

Plugging into the value function, we have

$$W(k_t) = \frac{\ln(1-\beta) + \ln(k_t)}{1-\beta} + \frac{\beta \ln(\beta)}{(1-\beta)^2} = V(k_t).$$

Thus, the Bellman Equation (11.5) holds.

In the following section, we will briefly show that the Bellman Equation holds in a general setting. That is, the solution to the initial problem solves the Bellman equation.

Moreover, the solution to the Bellman Equation is also a solution to the initial problem. Our discussions will be focused on infinite-horizon discrete-time models. In fact, dynamic programming is especially useful for when time is discrete (and there is uncertainty).

## 11.B. Dynamic Programming

We reformulate the initial problem into a **Sequence Problem**.

**Definition 11.B.1** (Sequence Problem)**.** The sequence problem is of the form:

$$V(x_0) = \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \tag{SP}$$

$$\text{s.t. } x_{t+1} \in \Gamma(x_t) \text{ for all } t = 0, 1, 2, ...$$

$$x_0 \in X \text{ given.}$$

**Example 11.1** (Life-cycle saving problem (infinite-horizon))**.** Formulating the previous life-cycle saving problem into a sequence problem, we have:

$$V(k_0) = \sup_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(k_t - k_{t+1})$$

$$\text{s.t. } k_{t+1} \in [0, k_t] \equiv \Gamma(k_t) \text{ for all } t = 0, 1, 2, ...$$

$$k_0 > 0 \text{ given.}$$

**Definition 11.B.2** (Bellman Equation)**.**

$$V(x_t) = \sup_{x_{t+1} \in \Gamma(x_t)} \{F(x_t, x_{t+1}) + \beta V(x_{t+1})\} \text{ for all } x_t \in X \tag{BE}$$

As mentioned before, Bellman equation expresses the value function as a combination of a flow payoff $F(x_t, x_{t+1})$ and a discounted continuation payoff $\beta V(x_{t+1})$. We call the time-invariant value function $V(\cdot)$ the solution to Bellman equation.[1]

We briefly show below that the value function defined by the sequence problem is also the solution to the Bellman equation and vice versa (with an additional condition $\lim_{n\to\infty} \beta^n V(x_n) = 0$ for any feasible $x$ sequences).[2]

[1] We haven't yet demonstrated that a solution $V(\cdot)$ exists.

[2] For this claim to hold, assumptions are needed to ensure that the sequence problem is well-defined. For the assumptions needed and a detailed proof, see chapter 4 of Stokey, Lucas & Prescott (1989).

1. A solution to the sequence problem is also a solution to Bellman equation.

$$\begin{aligned}
V(x_0) &= \sup_{\{x_{t+1}\in\Gamma(x_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) = \sup_{\{x_{t+1}\in\Gamma(x_t)\}_{t=0}^{\infty}} \Big(F(x_0, x_1) + \sum_{t=1}^{\infty} \beta^t F(x_t, x_{t+1})\Big) \\
&= \sup_{\{x_{t+1}\in\Gamma(x_t)\}_{t=0}^{\infty}} \Big(F(x_0, x_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} F(x_t, x_{t+1})\Big) \\
&= \sup_{x_1\in\Gamma(x_0)} \Big(F(x_0, x_1) + \beta \sup_{\{x_{t+1}\in\Gamma(x_t)\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_{t+1}, x_{t+2})\Big) \\
&= \sup_{x_1\in\Gamma(x_0)} \Big(F(x_0, x_1) + \beta V(x_1)\Big)
\end{aligned}$$

2. Under the condition $\lim_{n\to\infty} \beta^n V(x_n) = 0$ for any feasible $x$ sequences, a solution to Bellman equation is also a solution to the sequence problem.

$$\begin{aligned}
V(x_0) &= \sup_{x_1\in\Gamma(x_0)} \Big(F(x_0, x_1) + \beta V(x_1)\Big) \\
&= \sup_{x_1\in\Gamma(x_0)} \Big(F(x_0, x_1) + \beta \sup_{x_2\in\Gamma(x_1)} [F(x_1, x_2) + \beta V(x_2)]\Big) \\
&= \sup_{\{x_{t+1}\in\Gamma(x_t)\}_{t=0}^{1}} \Big(F(x_0, x_1) + \beta[F(x_1, x_2) + \beta V(x_2)]\Big) \\
&\quad \dots \\
&= \sup_{\{x_{t+1}\in\Gamma(x_t)\}_{t=0}^{n-1}} \Big(F(x_0, x_1) + \beta F(x_1, x_2) + ... + \beta^{n-1} F(x_{n-1}, x_n) + \beta^n V(x_n)\Big) \\
&\quad \dots \\
&= \sup_{\{x_{t+1}\in\Gamma(x_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) + \lim_{n\to\infty} \beta^n V(x_n) \\
&= \sup_{\{x_{t+1}\in\Gamma(x_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})
\end{aligned}$$

## 11.C. Solving Bellman equation

There are in general three methods to solve the Bellman equation:

- Guess and verify
- Iterate functional operator analytically
- Iterate functional operator numerically (We will not cover this method in this course.)

### 11.C.1. Guess and verify

Let us reconsider the infinite-horizon version of the life-cycle saving problem. Bellman equation of the problem states:

$$V(k_t) = \max_{k_{t+1}\in[0,k_t]} \{\ln(k_t - k_{t+1}) + \beta V(k_{t+1})\}.$$

The solution must be interior. We have the following two conditions.

1. FOC:
$$-\frac{1}{k_t - k_{t+1}} + \beta V'(k_{t+1}) = 0.$$
2. Envelope theorem:
$$V'(k_t) = \frac{1}{k_t - k_{t+1}}.$$

**Guess the value function** Guess that the value function takes the form:

$$V(k) = a + b\ln(k),$$

where $a$ and $b$ are constants to be determined. We try this form because the utility function is of the log form. Then, Bellman equation becomes:

$$a + b\ln(k_t) = \max_{k_{t+1}\in[0,k_t]} \Big\{ \ln(k_t - k_{t+1}) + \beta\Big(a + b\ln(k_{t+1})\Big)\Big\} \tag{11.7}$$

FOC and envelope theorem become:

$$-\frac{1}{k_t - k_{t+1}} + \frac{\beta b}{k_{t+1}} = 0 \implies k_{t+1} = \frac{\beta b}{\beta b + 1} k_t \tag{11.8}$$

$$\frac{b}{k_t} = \frac{1}{k_t - k_{t+1}} \implies k_{t+1} = \frac{b-1}{b} k_t \tag{11.9}$$

(11.8) and (11.9) implies $b = \frac{1}{1-\beta}$ and $k_{t+1} = \beta k_t$. Plugging this back into (11.7), we have

$$a + \frac{1}{1-\beta}\ln(k_t) = \ln((1-\beta)k_t) + \beta\Big(a + \frac{1}{1-\beta}\ln(\beta k_t)\Big) \iff a = \frac{\ln(1-\beta)}{1-\beta} + \frac{\beta\ln(\beta)}{(1-\beta)^2}.$$

Therefore,
$$V(k) = \frac{\ln(1-\beta)}{1-\beta} + \frac{\beta\ln(\beta)}{(1-\beta)^2} + \frac{1}{1-\beta}\ln(k)$$

is a solution to Bellman equation. Note that this solution is the same as the value function (11.6) we calculated previously.

**Guess the policy function** Alternatively, we could also guess the form of the policy function. Guess $k_{t+1} = \theta k_t$, where $\theta$ is a constant to be determined. Then envelope theorem implies

$$V'(k_t) = \frac{1}{(1-\theta)k_t}$$

Substitute $V'(k_{t+1}) = \frac{1}{(1-\theta)k_{t+1}}$ and then $k_{t+1} = \theta k_t$ into FOC

$$-\frac{1}{k_t - k_{t+1}} + \beta\frac{1}{(1-\theta)k_{t+1}} = 0 \implies -\frac{1}{(1-\theta)k_t} + \beta\frac{1}{(1-\theta)\theta k_t} = 0 \implies \theta = \beta.$$

We get the policy function $k_{t+1} = \beta k_t$, which implies $k_t = \beta^t k_0$. So, the value function is

$$\begin{aligned} V(k_0) &= \sum_{t=0}^{\infty} \beta^t \ln(k_t - k_{t+1}) = \sum_{t=0}^{\infty} \beta^t \ln(\beta^t k_0 - \beta^{t+1}k_0) \\ &= \sum_{t=0}^{\infty} \beta^t [t\ln(\beta) + \ln(1-\beta) + \ln(k_0)] \\ &= \frac{\beta\ln(\beta)}{(1-\beta)^2} + \frac{\ln(1-\beta)}{1-\beta} + \frac{1}{1-\beta}\ln(k_0) \end{aligned}$$

### 11.C.2. Iterate functional operator analytically

**How to do it** Still consider the infinite-horizon version of the life-cycle saving problem. Start with any initial guess, for example, $V_0(k) = 0$. Then the first iteration gives

$$V_1(k_t) = \max_{k_{t+1}\in[0,k_t]}\{\ln(k_t - k_{t+1}) + \beta V_0(k_{t+1})\} = \max_{k_{t+1}\in[0,k_t]}\{\ln(k_t - k_{t+1})\}.$$

The objective function is decreasing in $k_{t+1}$, so the optimal choice is $k_{t+1} = 0$. Then

$$V_1(k_t) = \ln(k_t).$$

The second iteration is

$$V_2(k_t) = \max_{k_{t+1}\in[0,k_t]}\{\ln(k_t - k_{t+1}) + \beta V_1(k_{t+1})\} = \max_{k_{t+1}\in[0,k_t]}\{\ln(k_t - k_{t+1}) + \beta\ln(k_{t+1})\}.$$

FOC gives

$$-\frac{1}{k_t - k_{t+1}} + \beta\frac{1}{k_{t+1}} = 0 \implies k_{t+1} = \frac{\beta}{1+\beta}k_t.$$

Then

$$V_2(k_t) = \ln(k_t - \frac{\beta}{1+\beta}k_t) + \beta\ln(\frac{\beta}{1+\beta}k_t) = \text{some constant} + (1+\beta)\ln(k_t)$$

The third iteration is

$$\begin{aligned} V_3(k_t) &= \max_{k_{t+1}\in[0,k_t]} \{\ln(k_t - k_{t+1}) + \beta V_2(k_{t+1})\} \\ &= \max_{k_{t+1}\in[0,k_t]} \Big\{ \ln(k_t - k_{t+1}) + \beta\Big[\text{some constant} + (1+\beta)\ln(k_{t+1})\Big]\Big\} \end{aligned}$$

FOC gives

$$-\frac{1}{k_t - k_{t+1}} + \beta\frac{1+\beta}{k_{t+1}} = 0 \implies k_{t+1} = \frac{\beta(1+\beta)}{1+\beta(1+\beta)}k_t.$$

Then

$$V_3(k_t) = \text{some constant } + (1+\beta+\beta^2)\ln(k_t)$$

Continuing iteration, eventually, we will obtain

$$V(k_t) = \text{some constant } + \frac{1}{1-\beta}\ln(k_t)$$

and

$$\begin{aligned} V(k_t) &= \max_{k_{t+1}\in[0,k_t]} \Big\{ \ln(k_t - k_{t+1}) + \beta V(k_{t+1})\Big\} \\ &= \max_{k_{t+1}\in[0,k_t]} \Big\{ \ln(k_t - k_{t+1}) + \beta\Big[\text{some constant} + \frac{1}{1-\beta}\ln(k_{t+1})\Big]\Big\} \end{aligned}$$

FOC gives

$$-\frac{1}{k_t - k_{t+1}} + \frac{\beta}{(1-\beta)k_{t+1}} = 0 \implies k_{t+1} = \beta k_t.$$

After obtaining the policy function, we could get the value function (See last section: Guess the policy function).

**Remark 1.** In this example, we have shown that $\lim_{n\to\infty} V_n \to V$ when $V_0(k) = 0$. In fact, we will always get convergence independent of the choice of $V_0$. The theory will be briefly discussed later.

The above iteration method could be described in a more convenient way. For any function $w : \mathbb{R}_+ \to \mathbb{R}$, we can define a new function $Bw : \mathbb{R}_+ \to \mathbb{R}$ by

$$(Bw)(k_t) = \max_{k_{t+1}\in[0,k_t]} \Big\{ \ln(k_t - k_{t+1}) + \beta w(k_{t+1})\Big\}.$$

When we use this notation, the previous method is equivalent to choosing a function $V_0$

and studying the sequence $\{V_n\}$ defined by $V_{n+1} = BV_n$ for $n = 0, 1, 2, ....$ The goal is to show that this sequence of functions converge to the limit function $V$ that satisfies

$$V(k_t) = \max_{k_{t+1}\in[0,k_t]} \Big\{ \ln(k_t - k_{t+1}) + \beta V(k_{t+1}) \Big\}. \tag{11.10}$$

Or equivalently, we could view $B$ as a mapping from some set of functions into itself. Then, what we are looking for is a fixed point of the mapping $B$, that is, a function $V$ that satisfies $V = BV$. The operator $B$ is called **Bellman operator**.

In a general setting, Bellman operator is defined as follows:

$$(Bw)(x_t) = \sup_{x_{t+1}\in\Gamma(x_t)} \{F(x_t, x_{t+1}) + \beta w(x_{t+1})\} \text{ for all } x_t \in X \tag{BE}$$

What we do is to pick some $w$ and iterate $B^n w$ until convergence:

$$\begin{aligned}
(Bw)(x_t) &= \sup_{x_{t+1}\in\Gamma(x_t)} \{F(x_t, x_{t+1}) + \beta w(x_{t+1})\} \\
(B(Bw))(x_t) &= \sup_{x_{t+1}\in\Gamma(x_t)} \{F(x_t, x_{t+1}) + \beta (Bw)(x_{t+1})\} \\
(B(B^2w))(x_t) &= \sup_{x_{t+1}\in\Gamma(x_t)} \{F(x_t, x_{t+1}) + \beta (B^2w)(x_{t+1})\} \\
&\dots \\
(B(B^nw))(x_t) &= \sup_{x_{t+1}\in\Gamma(x_t)} \{F(x_t, x_{t+1}) + \beta (B^nw)(x_{t+1})\} \\
&\dots
\end{aligned}$$

(Uniform) convergence of a sequence of functions is defined by convergence in sup-norm.

**Why it works** We will briefly discuss why the sequence always converges. The short answer is: $B$ is a contraction mapping.

**Definition 11.C.1** (Contraction mapping)**.** Let $(S, \rho)$ be a metric space and $T : S \to S$ be a function mapping $S$ into itself. $T$ is a **contraction mapping** (with **modulus** $\beta$) if for some $\beta \in (0, 1)$, $\rho(Tx, Ty) \leq \beta\rho(x, y)$, for all $x, y \in S$.

In plain words, $T$ is a contraction mapping if operating $T$ on any two elements in $S$ moves them strictly closer to each other.

For our result, we need the following two results:

1. Contraction Mapping Theorem (Theorem 11.1): a fixed point theorem
2. Blackwell's sufficient conditions (Theorem 11.2): sufficient conditions for an operator to be a contraction mapping

**Theorem 11.1** (Contraction Mapping Theorem (Stokey, Lucas & Prescott Theorem 3.2))**.** *If $(S, \rho)$ is a complete metric space and $T : S \to S$ is a contraction mapping with modulus $\beta$, then*

*a. $T$ has exactly one fixed point $v$ in $S$, and*

*b. for any $v_0 \in S$, $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$, $n = 0, 1, 2, ....$*

Sketch of proof: To prove (a), we

1. find a candidate for $v$;
2. show that $Tv = v$;
3. show that for any other element $\hat{v} \in S$, $T\hat{v} \neq \hat{v}$.

For 1, define the iterate of $T$, the mapping $\{T^n\}$ by $T^0 x = x$ and $T^n x = T(T^{n-1}x)$, $n = 1, 2, ....$ Choose $v_0 \in S$ and define the sequence $\{v_n\}_{n=0}^{\infty}$ by $v_{n+1} = Tv_n$ so that $v_n = T^n v_0$. Show that $v_n$ is a Cauchy sequence. (This is done using 1) $T$ being a contraction mapping, and 2) triangle inequality.) Then $S$ is complete, we have $v_n \to v \in S$.

For 2, for all $n$ and all $v_0 \in S$,

$$\rho(Tv, v) \leq \rho(Tv, T^n v_0) + \rho(T^n v_0, v) \leq \beta\rho(v, T^{n-1}v_0) + \rho(T^n v_0, v)$$

Both $\rho(v, T^{n-1}v_0)$ and $\rho(T^n v_0, v)$ converge to 0 as $n \to \infty$. Hence $\rho(Tv, v) = 0$, i.e., $Tv = v$.

For 3, suppose to the contrary, $\hat{v} \neq v$ is another solution. Then

$$\rho(\hat{v}, v) = \rho(T\hat{v}, Tv) \leq \beta\rho(\hat{v}, v) \tag{11.11}$$

Since $\beta \in (0, 1)$, (11.11) never holds.

To prove (b), observe that for any $n \geq 1$,

$$\rho(T^n v_0, v) = \rho(T(T^{n-1}v_0), Tv) \leq \beta\rho(T^{n-1}v_0, v).$$

So (b) follows by induction.

**Theorem 11.2** (Blackwell's sufficient conditions for a contraction (Stokey, Lucas & Prescott Theorem 3.3))**.** *Let* $X \subseteq \mathbb{R}^l$*, and let* $B(X)$ *be a space of bounded functions* $f : X \to \mathbb{R}$*, with the sup norm. Let* $T : B(X) \to B(X)$ *be an operator satisfying*

*a. (monotonicity)* $f, g \in B(X)$ *and* $f(x) \leq g(x)$*, for all* $x \in X$*, implies* $(Tf)(x) \leq (Tg)(x)$*, for all* $x \in X$*;*

*b. (discounting) there exists some* $\beta \in (0, 1)$ *such that*

$$[T(f + a)](x) \leq (Tf)(x) + \beta a, \text{ all } f \in B(X), a \geq 0, x \in X.$$

*[Here* $(f + a)(x)$ *is the function defined by* $(f + a)(x) = f(x) + a$*.]*

*Then* $T$ *is a contraction with modulus* $\beta$*.*

**Proof.** If $f(x) \leq g(x)$ for all $x \in X$, we write $f \leq g$. For any $f, g \in B(X)$, $f \leq g + ||f - g||$. Then properties (a) and (b) imply that

$$Tf \leq T(g + ||f - g||) \leq Tg + \beta ||f - g|| \implies Tf - Tg \leq \beta ||f - g||.$$

Similarly,

$$Tg \leq T(f + ||f - g||) \leq Tf + \beta ||f - g|| \implies Tg - Tf \leq \beta ||f - g||.$$

Combining the two inequalities,

$$||Tf - Tg|| \leq \beta ||f - g||.$$

**Remark 2.** Blackwell's sufficient conditions are only sufficient but not necessary: some contraction mappings do not satisfy these sufficient conditions.

**Example 11.2.** Check Blackwell sufficient conditions for the life-cycle saving problem:

$$(Bw)(k_t) = \max_{k_{t+1} \in [0, k_t]} \Big\{ \ln(k_t - k_{t+1}) + \beta w(k_{t+1}) \Big\}$$

Suppose $v \leq w$. Let $k^*_{t+1}$ be the optimal choice when continuation value function is $v$.

1. For (a)

$$
\begin{aligned}
(Bv)(k_t) &= \max_{k_{t+1}\in[0,k_t]} \Big\{ \ln(k_t - k_{t+1}) + \beta v(k_{t+1}) \Big\} = \ln(k_t - k^*_{t+1}) + \beta v(k^*_{t+1}) \\
&\leq \ln(k_t - k^*_{t+1}) + \beta w(k^*_{t+1}) \leq \max_{k_{t+1}\in[0,k_t]} \Big\{ \ln(k_t - k_{t+1}) + \beta w(k_{t+1}) \Big\} \\
&= (Bw)(k_t)
\end{aligned}
$$

2. For (b)

$$
\begin{aligned}
[B(w+a)](k_t) &= \max_{k_{t+1}\in[0,k_t]} \Big\{ \ln(k_t - k_{t+1}) + \beta(w+a)(k_{t+1}) \Big\} \\
&= \max_{k_{t+1}\in[0,k_t]} \Big\{ \ln(k_t - k_{t+1}) + \beta[w(k_{t+1}) + a] \Big\} \\
&= \max_{k_{t+1}\in[0,k_t]} \Big\{ \ln(k_t - k_{t+1}) + \beta[w(k_{t+1})] \Big\} + \beta a \\
&= (Bw)(k_t) + \beta a.
\end{aligned}
$$

Therefore, by Blackwell's sufficient conditions (Theorem 11.2), $B$ is a contraction mapping. And by Contraction Mapping Theorem (Theorem 11.1), $B$ has a unique fixed point, which could be reached from any initial point.

**Remark 3.** This result implies that the Bellman equation has a unique solution.

## 11.D. Examples

### 11.D.1. Example 1: Optimal growth model

**Finite-horizon, backward induction** Consider the following social planner's problem:

$$
\max_{\substack{\{c_t\}_{t=0}^T \\ \{k_t\}_{t=1}^T}} \sum_{t=0}^{T} \beta^t \ln(c_t)
$$

$$
\text{s.t. } c_t + k_{t+1} = k_t^\alpha \text{ for all } t = 0, ..., T
$$

$k_0 > 0$ is given and the terminal capital $k_{T+1} = 0$. In the model, $f(k_t) = k_t^\alpha$ is the production function, and capital is fully depreciated, i.e., $\delta = 1$.

We will apply dynamic programming to solve the model when $T = 2$. The method of solving the problem extends to all finite $T$.

$T = 2$ problem is:

$$\max_{\substack{c_0,c_1,c_2 \\ k_1,k_2}} \ln(c_0) + \beta \ln(c_1) + \beta^2 \ln(c_2) \tag{11.12}$$

$$\text{s.t. } c_t + k_{t+1} = k_t^\alpha \text{ for all } t = 0, 1, 2.$$

$k_0 > 0$ given and $k_3 = 0$.

We solve the problem by *Backward Induction*.

1. At $t = 2$, since $k_3 = 0$, we have $c_2 = k_2^\alpha$. So the value function is

$$V_2(k_2) = \alpha \ln(k_2).$$

2. At $t = 1$, we have $c_1 = k_1^\alpha - k_2$. The Bellman equation is

$$V_1(k_1) = \max_{k_2 \in [0, k_1^\alpha]} \{\ln(k_1^\alpha - k_2) + \beta V_2(k_2)\} = \max_{k_2 \in [0, k_1^\alpha]} \{\ln(k_1^\alpha - k_2) + \alpha\beta \ln(k_2)\}$$

FOC gives

$$-\frac{1}{k_1^\alpha - k_2} + \frac{\alpha\beta}{k_2} = 0 \implies k_2 = \frac{\alpha\beta}{1 + \alpha\beta} k_1^\alpha \tag{11.13}$$

So the value function is

$$V_1(k_1) = \ln(k_1^\alpha - \frac{\alpha\beta}{1 + \alpha\beta} k_1^\alpha) + \alpha\beta \ln(\frac{\alpha\beta}{1 + \alpha\beta} k_1^\alpha) = \text{some constant} + \alpha(1 + \alpha\beta) \ln(k_1).$$

3. At $t = 0$, we have $c_0 = k_0^\alpha - k_1$. The Bellman equation is

$$\begin{aligned} V_0(k_0) &= \max_{k_1 \in [0, k_0^\alpha]} \{\ln(k_0^\alpha - k_1) + \beta V_1(k_1)\} \\ &= \max_{k_1 \in [0, k_0^\alpha]} \{\ln(k_0^\alpha - k_1) + \beta[\text{some constant} + \alpha(1 + \alpha\beta) \ln(k_1)]\} \end{aligned}$$

FOC gives

$$-\frac{1}{k_0^\alpha - k_1} + \frac{\alpha\beta(1 + \alpha\beta)}{k_1} = 0 \implies k_1 = \frac{\alpha\beta(1 + \alpha\beta)}{1 + \alpha\beta(1 + \alpha\beta)} k_0^\alpha \tag{11.14}$$

The problem is fully solved: equation (11.14) defines $k_1$, and substituting it into equation (11.13) gives $k_2$ as a function of $k_0$. We could also recover $c_0$, $c_1$ and $c_2$ from the constraints.

**Infinite-horizon** Consider infinite-horizon version of the above social planner's problem:

$$\max_{\substack{\{c_t\}_{t=0}^{\infty} \\ \{k_t\}_{t=1}^{\infty}}} \sum_{t=0}^{\infty} \beta^t \ln(c_t) \tag{11.15}$$

$$\text{s.t. } c_t + k_{t+1} = k_t^{\alpha} \text{ for all } t = 0, 1, 2, ...$$

$k_0 > 0$ is given.

We apply the guess and verify method to solve the problem.

The Bellman equation is

$$V(k_t) = \max_{k_{t+1} \in [0, k_t^{\alpha}]} \{\ln(k_t^{\alpha} - k_{t+1})) + \beta V(k_{t+1})\}.$$

The solution must be interior. We have the following two conditions.

1. FOC:
$$-\frac{1}{k_t^{\alpha} - k_{t+1}} + \beta V'(k_{t+1}) = 0.$$
2. Envelope theorem:
$$V'(k_t) = \frac{\alpha k_t^{\alpha-1}}{k_t^{\alpha} - k_{t+1}}.$$

**Guess the value function** Guess that the value function takes the form:

$$V(k) = a + b\ln(k),$$

where $a$ and $b$ are constants to be determined. We try this form because the utility function is of the log form. Then, Bellman equation becomes:

$$a + b\ln(k_t) = \max_{k_{t+1} \in [0, k_t]} \left\{ \ln(k_t^{\alpha} - k_{t+1}) + \beta\Big(a + b\ln(k_{t+1})\Big)\right\} \tag{11.16}$$

FOC and envelope theorem become:

$$-\frac{1}{k_t^{\alpha} - k_{t+1}} + \frac{\beta b}{k_{t+1}} = 0 \implies k_{t+1} = \frac{\beta b}{\beta b + 1} k_t^{\alpha} \tag{11.17}$$

$$\frac{b}{k_t} = \frac{\alpha k_t^{\alpha-1}}{k_t^{\alpha} - k_{t+1}} \implies k_{t+1} = \frac{b - \alpha}{b} k_t^{\alpha} \tag{11.18}$$

(11.17) and (11.18) implies $b = \frac{\alpha}{1-\alpha\beta}$ and $k_{t+1} = \alpha\beta k_t^{\alpha}$. Plugging this back into (11.16), we could recover $a$ and accordingly the value function $V(k)$.

**Guess the policy function** Alternatively, we could also guess the form of the policy function. Guess $k_{t+1} = \theta f(k_t) = \theta k_t^\alpha$, where $\theta$ is a constant to be determined. Then envelope theorem implies

$$V'(k_t) = \frac{\alpha}{(1-\theta)k_t}$$

Substitute $V'(k_{t+1}) = \frac{\alpha}{(1-\theta)k_{t+1}}$ and then $k_{t+1} = \theta k_t^\alpha$ into FOC

$$-\frac{1}{k_t^\alpha - k_{t+1}} + \beta \frac{\alpha}{(1-\theta)k_{t+1}} = 0 \implies -\frac{1}{(1-\theta)k_t^\alpha} + \beta \frac{\alpha}{(1-\theta)\theta k_t^\alpha} = 0 \implies \theta = \alpha\beta.$$

We get the policy function $k_{t+1} = \alpha\beta k_t^\alpha$, which implies $k_t = (\alpha\beta)^{\frac{1-\alpha^t}{1-\alpha}} k_0^{\alpha^t}$. So, the value function is

$$\begin{aligned} V(k_0) &= \sum_{t=0}^{\infty} \beta^t \ln((1-\alpha\beta)k_t^\alpha) = \sum_{t=0}^{\infty} \left[\beta^t \ln(1-\alpha\beta) + \alpha\beta^t \ln(k_t)\right] \\ &= \frac{\ln(1-\alpha\beta)}{1-\beta} + \alpha \sum_{t=0}^{\infty} \left[\beta^t \Big(\frac{1-\alpha^t}{1-\alpha} \ln(\alpha\beta) + \alpha^t \ln(k_0)\Big)\right] \\ &= \frac{\ln(1-\alpha\beta)}{1-\beta} + \alpha \sum_{t=0}^{\infty} \left[\beta^t \Big(\frac{1-\alpha^t}{1-\alpha} \ln(\alpha\beta) + \alpha^t \ln(k_0)\Big)\right] \\ &= \frac{\ln(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta \ln(\alpha\beta)}{(1-\beta)(1-\alpha\beta)} + \frac{\alpha \ln(k_0)}{1-\alpha\beta} \end{aligned}$$

**Iterate functional operator analytically** We could obtain the same result by iterating functional operator analytically. For example, try the initial guess $V_0(k_t) = 0$.

**Stochastic growth** Dynamic programming is also applicable to stochastic problems. The social planner's problem is modified to be the following:

$$\max_{\substack{\{c_t(z_t)\}_{t=0}^{\infty} \\ \{k_t(z_t)\}_{t=1}^{\infty}}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \ln(c_t) \tag{11.19}$$

$$\text{s.t. } c_t + k_{t+1} = z_t k_t^\alpha \text{ for all } t = 0, 1, 2, ...$$

$k_0 > 0$ is given. $\{z_t\}$ is a sequence of independently and identically distributed random variables with $\mathbb{E}_0(\ln(z_t)) = \mu$. At the beginning of period $t$, the exogenous shock $z_t$ is realized. Thus when making period $t$ decision, the social planner knows the pair $(k_t, z_t)$ and accordingly the current output $z_t k_t^\alpha$. The pair $(k_t, z_t)$ is called the state of the economy. Note that now the solution is expressed in terms of *contingency plans*, that is,

$c_t$ and $k_{t+1}$ are functions of $z_t$.

The problem could still be equivalently expressed using recursive formulation. The Bellman equation is:

$$V(k_t, z_t) = \max_{k_{t+1} \in [0, z_t k_t^\alpha]} \{\ln(z_t k_t^\alpha - k_{t+1})) + \beta \mathbb{E}_t V(k_{t+1}, z_{t+1})\}.$$

Then FOC and envelope theorem give

1. FOC:
$$-\frac{1}{z_t k_t^\alpha - k_{t+1}} + \beta \mathbb{E}_t \frac{\partial V(k_{t+1}, z_{t+1})}{\partial k_{t+1}} = 0.$$
2. Envelope theorem:
$$\frac{\partial V(k_t, z_t)}{\partial k_t} = \frac{\alpha z_t k_t^{\alpha-1}}{z_t k_t^\alpha - k_{t+1}}.$$

To solve the problem, similar to the deterministic model, we guess and verify.

**Guess the value function** Guess that the value function takes the form:

$$V(k) = a + b\ln(k) + c\ln(z),$$

where $a$, $b$ and $c$ are constants to be determined. Then, Bellman equation becomes:

$$a + b\ln(k_t) + c\ln(z_t) = \max_{k_{t+1} \in [0, k_t]} \left\{ \ln(z_t k_t^\alpha - k_{t+1}) + \beta \mathbb{E}_t \Big( a + b\ln(k_{t+1}) + c\ln(z_{t+1}) \Big) \right\} \quad (11.20)$$

FOC and envelope theorem become:

$$-\frac{1}{z_t k_t^\alpha - k_{t+1}} + \frac{\beta b}{k_{t+1}} = 0 \implies k_{t+1} = \frac{\beta b}{\beta b + 1} z_t k_t^\alpha \quad (11.21)$$

$$\frac{b}{k_t} = \frac{\alpha z_t k_t^{\alpha-1}}{z_t k_t^\alpha - k_{t+1}} \implies k_{t+1} = \frac{b - \alpha}{b} z_t k_t^\alpha \quad (11.22)$$

(11.21) and (11.22) implies $b = \frac{\alpha}{1-\alpha\beta}$ and $k_{t+1} = \alpha\beta z_t k_t^\alpha$. Plugging this back into (11.20), we could recover $a$ and $c$:

$$a = \frac{\ln(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta\ln(\alpha\beta)}{(1-\beta)(1-\alpha\beta)} + \frac{\beta\mu}{(1-\beta)(1-\alpha\beta)}$$

$$c = \frac{1}{1-\alpha\beta}$$

and accordingly the value function $V(k)$.

**Guess the policy function** Alternatively, we could also guess the form of the policy function:

$$k_{t+1} = \theta f(k_t) = \theta z_t k_t^\alpha,$$

where $\theta$ is a constant to be determined. Then envelope theorem gives

$$\frac{\partial V(k_t, z_t)}{\partial k_t} = \frac{\alpha z_t k_t^{\alpha-1}}{z_t k_t^\alpha - \theta z_t k_t^\alpha} = \frac{\alpha}{(1-\theta)k_t}.$$

Substituting into FOC

$$\frac{1}{z_t k_t^\alpha - k_{t+1}} = \beta \mathbb{E}_t \frac{\alpha}{(1-\theta)k_{t+1}} \implies \frac{1}{z_t k_t^\alpha - \theta z_t k_t^\alpha} = \beta \mathbb{E}_t \frac{\alpha}{(1-\theta)\theta z_t k_t^\alpha} \implies \theta = \alpha\beta.$$

We get the policy function $k_{t+1} = \alpha\beta z_t k_t^\alpha$. The value function could be recovered using the policy function. To do this, we express the value function as

$$V(k_0) = \sum_{t=0}^{\infty} \beta^t \ln((1-\alpha\beta) z_t k_t^\alpha).$$

[The calculation is quite involved here.]

### 11.D.2. Example 2: Job market search (Dixit Example 1 + unemployment compensation)

There is a whole spectrum of jobs paying different wages in the economy. Denote the wage offer by $w$. The cumulative distribution function, the probability that a randomly selected job pays $w$ or less, is $\Phi(w)$. The corresponding density function is $\phi(w) = \Phi'(w)$. A worker must engage in search to find out how much a particular job pays. Each period, an unemployed worker draws a wage offer $w$. He could either accept or reject the offer. If the offer is rejected, then the worker stays unemployed and waits until the next period to draw another wage offer. The worker receives unemployment compensation $c$ for each of the unemployed period. The discount factor is $\beta$.

**Analysis.** The Bellman equation for the worker's problem is

$$V(w) = \max\{c + \beta \int_0^\infty V(w') \mathrm{d}\Phi(w'), \frac{w}{1-\beta}\}.$$

We define $w^* = \min\{w | \frac{w}{1-\beta} \geq c + \beta \int_0^\infty V(w')\mathrm{d}\Phi(w')\}$. That is, $\frac{w^*}{1-\beta} = c+\beta \int_0^\infty V(w')\mathrm{d}\Phi(w')$. Then $w^*$ is the unique threshold value such that the worker decides to take the offer whenever $w \geq w^*$ for the first time. To see this, consider another the wage offer $\hat{w} \geq w^*$. The worker must accept $\hat{w}$:

$$c + \beta \int_0^\infty V(w')\mathrm{d}\Phi(w') = \frac{w^*}{1-\beta} \leq \frac{\hat{w}}{1-\beta}.$$

Since the worker obtains a higher payoff accepting $\hat{w}$ compared to staying unemployed, the worker will accept $\hat{w}$. Therefore, the value function satisfies

$$V(w) = \begin{cases} \frac{w}{1-\beta} & \text{if } w \geq w^* \\ \frac{w^*}{1-\beta} = c + \beta \int_0^\infty V(w')\mathrm{d}\Phi(w') & \text{if } w < w^* \end{cases}$$

Evaluating $w^*$, we have

$$\begin{aligned}
&\frac{w^*}{1-\beta} = c + \beta\Big[\int_{w^*}^\infty \frac{w}{1-\beta}\mathrm{d}\Phi(w) + \int_0^{w^*} \frac{w^*}{1-\beta}\mathrm{d}\Phi(w)\Big] \\
\Longrightarrow\ & w^* = c(1-\beta) + \beta\Big[\int_{w^*}^\infty w\mathrm{d}\Phi(w) + \int_0^{w^*} w^*\mathrm{d}\Phi(w)\Big] = c(1-\beta) + \beta\Big[w^* + \int_{w^*}^\infty (w - w^*)\mathrm{d}\Phi(w)\Big] \\
\Longrightarrow\ & w^* = c + \frac{\beta}{1-\beta}\int_{w^*}^\infty (w - w^*)\mathrm{d}\Phi(w)
\end{aligned}$$

Implications:

1. $w^* > c$ for non-degenerate distributions.
2. An increase in $c$ leads to an increase in $w^*$.

### 11.D.3. Example 3: Saving under uncertainty (Dixit Example 2)

Consider a consumer with wealth $W$ that earns a *random* total return (principal plus interest) of $r$ per period, and no other income. That is, starting period $t$ with wealth $W_t$, if the consumer consumes $C_t$ and saves $W_t - C_t$, his random wealth at the start of the next period will be $W_{t+1} = r_{t+1}(W_t - C_t)$. Note that $r_{t+1}$ is not realized when making the consumption decision. Consumption of $C_t$ in any period gives him utility

$$U(C) = \frac{C^{1-\varepsilon}}{(1-\varepsilon)}, \text{ with } \varepsilon > 0.$$

The discount factor is $\beta$.

**Analysis.** Unlike the previous models, here we could not choose $W_{t+1}$ since $r_{t+1}$ is random and unknown when making the consumption decision. We could still apply the dynamic programming approach. The Bellman equation is

$$V(W_t) = \max_{C_t \in [0, W_t]} \Big\{ \frac{C_t^{1-\varepsilon}}{1-\varepsilon} + \beta \mathbb{E}_t (V(r_{t+1}(W_t - C_t))) \Big\}.$$

Guess the value function

$$V(W_t) = \frac{A W_t^{1-\varepsilon}}{1-\varepsilon},$$

where $A$ is a constant to be determined. The guess is also based on the utility form. Then the Bellman equation becomes:

$$\frac{A W_t^{1-\varepsilon}}{1-\varepsilon} = \max_{C_t \in [0, W_t]} \Big\{ \frac{C_t^{1-\varepsilon}}{1-\varepsilon} + \beta \mathbb{E}_t \Big( \frac{A (r_{t+1}(W_t - C_t))^{1-\varepsilon}}{1-\varepsilon} \Big) \Big\}$$
$$\Longrightarrow \frac{A W_t^{1-\varepsilon}}{1-\varepsilon} = \max_{C_t \in [0, W_t]} \Big\{ \frac{C_t^{1-\varepsilon}}{1-\varepsilon} + \frac{A\beta (W_t - C_t)^{1-\varepsilon}}{1-\varepsilon} \mathbb{E}_t [(r_{t+1})^{1-\varepsilon}] \Big\}$$

FOC implies

$$C_t^{-\varepsilon} - (W_t - C_t)^{-\varepsilon} A\beta \mathbb{E}_t [(r_{t+1})^{1-\varepsilon}] = 0 \implies \frac{C_t}{W_t} = \frac{1}{1 + (A\beta \mathbb{E}_t [(r_{t+1})^{1-\varepsilon})]^{1/\varepsilon}} \tag{11.23}$$

Envelope theorem gives

$$A W_t^{-\varepsilon} = (W_t - C_t)^{-\varepsilon} A\beta \mathbb{E}_t [(r_{t+1})^{1-\varepsilon}] \implies \frac{C_t}{W_t} = 1 - (\beta \mathbb{E}_t [(r_{t+1})^{1-\varepsilon})]^{1/\varepsilon} \tag{11.24}$$

The constant $A$ could be found by (11.23) and (11.24):

$$A^{1/\varepsilon} [1 - (\beta \mathbb{E}_t [(r_{t+1})^{1-\varepsilon})]^{1/\varepsilon}] = 1.$$

The condition for the existence of $A$ is $\beta \mathbb{E}_t [(r_{t+1})^{1-\varepsilon}) < 1$.