Chapter 3. Extensions and Generalizations

In this Chapter, we will learn several extentions and generalizations to the two-variable, one-constraint *Lagrange's Theorem* we learned in Chapter 2, namely,

(i) Allowing for more variables and constraints;

(ii) Including non-negative variables;

(iii) Adding inequality constraints (*Kuhn-Tucker Theorem*).

3.A. More variables and constraints

Recall *Lagrange's Theorem* we learned in Chapter 2:

Theorem 2.1 (Lagrange's Theorem)**.** *Suppose x is a two-dimensional vector, c is a scalar, and F and G functions taking scalar values. Suppose x^* solves the following maximization problem:*

$$\max_x F(x)$$

$$s.t. \;\; G(x) = c,$$

and the constraint qualification holds, that is, if $G_j(x^*) \neq 0$ for at least one j. Define

$$\mathcal{L}(x, \lambda) = F(x) + \lambda \left[c - G(x)\right]. \tag{2.10}$$

Then there is a value of λ such that

$$\mathcal{L}_j(x^*, \lambda) = 0 \; for \; j = 1, 2 \qquad \mathcal{L}_\lambda(x^*, \lambda) = 0. \tag{2.11}$$

In Theorem 2.1, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and we only have one constraint: $G(x) = c$. In this section, we will extend the theorem to n choice variables $x = (x_1, x_2, ..., x_n)^T$, and m constraints:[1]

$$G^i(x) = c_i, \quad i = 1, 2, ..., m.$$

[1]Here, the superscript on $G(x)$ denotes the constraint number. For instance, $G^i(x) = c_i$ denotes the i^{th} constraint. Please remember that we used the subscript on $G(x)$ to denote partial derivatives. For instance, $G_j(x) = \partial G(x)/\partial x_j$.

Assumption. $m < n$.

This assumption is to ensure that the maximization problem is solvable and interesting. If $m = n$, there are exactly the same number of equality constraints as there are variables. The variables could be solved solely using the constraints and the maximization problem would become trivial. If $m > n$, the constraints themselves could be mutually inconsistent, leading to non-existence of solutions.

To extend the Lagrange's method, we define λ_i as the Lagrange multiplier for each constraint, and the we could write the Lagrangian:

$$\mathcal{L}(x_1, ..., x_n, \lambda_1, ..., \lambda_m) = F(x_1, ..., x_n) + \sum_{i=1}^{m} \lambda_i \left[c_i - G^i(x_1, ..., x_n) \right]. \tag{3.1}$$

First-order necessary conditions are

$$\mathcal{L}_j = \partial \mathcal{L} / \partial x_j = F_j(x_1, ..., x_n) - \sum_{i=1}^{m} \lambda_i G^i_j(x_1, ..., x_n) = 0 \text{ for } j = 1, 2, ..., n; \tag{3.2}$$

$$\mathcal{L}_{\lambda_i} = \partial \mathcal{L} / \partial \lambda_i = c_i - G^i(x) = 0 \text{ for } i = 1, 2, ..., m. \tag{3.3}$$

We have $(n+m)$ equations in (3.2) and (3.3) to solve for $(n+m)$ variables $x_1^*, x_2^*, ..., x_n^*, \lambda_1, ..., \lambda_m$.

Vector-Matrix Form. There is nothing conceptually new. It is only introduced to make the equations look neat. First, you need to be familiarized with the following notations:[2]

$$G(x) = \begin{pmatrix} G^1(x) \\ \vdots \\ G^m(x) \end{pmatrix}; \quad c = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}; \quad \lambda = \begin{pmatrix} \lambda_1, ..., \lambda_m \end{pmatrix};$$

$$F_x(x) = \begin{pmatrix} F_1(x), ..., F_n(x) \end{pmatrix}; \quad G^i_x(x) = \begin{pmatrix} G^i_1(x), ..., G^i_n(x) \end{pmatrix};$$

$$G_x(x) = \begin{pmatrix} G^1_x(x) \\ G^2_x(x) \\ \vdots \\ G^m_x(x) \end{pmatrix} = \begin{bmatrix} G^1_1(x) & \dots & G^1_n(x) \\ G^2_1(x) & \dots & G^2_n(x) \\ \vdots & \ddots & \vdots \\ G^m_1(x) & \dots & G^m_n(x) \end{bmatrix}.$$

[2] We adopt the convention that when the argument of a function is a column vector, the vector of partial derivatives is a row vector, and vice versa. See $F_x(x)$ and $G^i_x(x)$.

With the new notations, the Lagrangian (3.1) could be written as

$$\mathcal{L}(x,\lambda) = F(x) + \lambda\left[c - G(x)\right] \tag{3.4}$$

First-order necessary conditions (3.2) and (3.3) could be written as

$$\mathcal{L}_x(x^*,\lambda) = F_x(x) - \lambda G_x(x) = 0, \tag{3.5}$$

$$\mathcal{L}_\lambda(x^*,\lambda) = c - G(x) = 0. \tag{3.6}$$

Constraint Qualification. In Chapter 2, we have learned that for two-variable, one-constraint case, to ensure the validity of the first-order necessary conditions, we need to check *Constraint Qualification.* We also learned that the condition is $(G_1(x^*), G_2(x^*))$ being a non-zero vector. For n-variable, m-constraint cases, *Constraint Qualification* is also required. The condition is that the matrix $G_x(x^*)$ should not have any singularity. That is, $G_x^i(x^*)$'s should be linearly independent, or $G_x(x^*)$ should have rank m.[3]
Again, in practice, failure of *Constraint Qualification* is rarely a problem. However, you should be alerted and check *Constraint Qualification* if you find standard methods problematic. Failure of *Constraint Qualification* could usually be circumvented by writing the algebriac form of the constraints differently.[4]

Now, we are ready to summarize the generalized *Lagrange's Theorem* for n varaibles and m constraints.

Theorem 3.1 (Lagrange's Theorem)**.** *Suppose x is a n-dimensional vector, c an m-dimensional vector, F a function taking scalar values, G a function taking m-dimensional vector values, with $m < n$. Suppose x^* solves the following maximization problem:*

$$\max_x F(x)$$

$$s.t. \;\; G(x) = c,$$

[3] Formal proofs are not required and will not be discussed in this course.
[4] See Chapter 2 Section 2.C.

and the constraint qualification holds, i.e., rank $G_x(x^*) = m$*. Define*

$$\mathcal{L}(x, \lambda) = F(x) + \lambda \left[c - G(x) \right], \tag{3.4}$$

where λ *is an* m*-dimensional row vector. Then there is a value of* λ *such that*

$$\mathcal{L}_x(x^*, \lambda) = 0, \tag{3.5}$$

$$\mathcal{L}_\lambda(x^*, \lambda) = 0. \tag{3.6}$$

3.B. Non-negative variables

Suppose that x_j must be non-negative to make economic sense. If the optimum x^* happens to be $x_j^* > 0$ for all j, then what we learned in Section 3.A continues to hold. However, if it is not true, say if $x_1^* = 0$, then only one side of the arbitrage argument would apply. More specifically, we can only consider infinitesimal changes dx for which $dx_1 > 0$. Therefore, when $x_1^* = 0$, Condition (3.2) is modified as[5]

$$\mathcal{L}_1(x^*, \lambda) = F_1(x^*) - \sum_{i=1}^{m} \lambda_i G_1^i(x^*) \leq 0. \tag{3.7}$$

Therefore,

i. when $x_j^* > 0$, (3.2) $\mathcal{L}_j(x^*, \lambda) = 0$ holds;

ii. when $x_j^* = 0$, (3.7) $\mathcal{L}_j(x^*, \lambda) \leq 0$ holds.

In other words, for every j

$$\mathcal{L}_j(x^*, \lambda) \leq 0 \text{ and } x_j^* \geq 0 \tag{3.8}$$

with at least one holding with equality.[6] The requirement that at least one inequalities hold with equality could be equivalently written as

$$x_j^* \mathcal{L}_j(x^*, \lambda) = 0,$$

and is called *complementary slackness*: one inequality complements the slackness in the other.[7]

[5]This could be viewed as generalization of Equation (1.4) in Chapter 1.
[6]This qualification rules out the case when both expressions hold with inequality.
[7]An inequality is called *binding* if it holds with equality; and *slack* if it holds with strict inequality.

We use vector-matrix form for simple exposition. Note that for a vector x:

(i) $x \geq 0$ means that $x_j \geq 0$ for all j;

(ii) $x > 0$ means that $x_j \geq 0$ for all j and at least one of $x_j > 0$;

(iii) $x \gg 0$ means that $x_j > 0$ for all j.

Using the new notation, (3.8) becomes

$$\mathcal{L}_x(x^*, \lambda) \leq 0 \text{ and } x^* \geq 0, \text{ with complementary slackness}^8. \tag{3.8}$$

The result is formally stated in Theorem 3.2 below:

Theorem 3.2 (Lagrange's Theorem with Non-Negative Variables)**.** *Suppose x is a n-dimensional vector, c an m-dimensional vector, F a function taking scalar values, G a function taking m-dimensional vector values, with $m < n$. Suppose x^* solves the following maximization problem:*

$$\max_x F(x)$$
$$s.t.\ G(x) = c \text{ and } x \geq 0,$$

and the constraint qualification holds, i.e., rank $G_x(x^*) = m$. Define

$$\mathcal{L}(x, \lambda) = F(x) + \lambda \left[c - G(x)\right], \tag{3.4}$$

where λ is an m-dimensional row vector. Then there is a value of λ such that

$$\mathcal{L}_x(x^*, \lambda) \leq 0, \ x^* \geq 0, \ \textit{with complementary slackness}, \tag{3.8}$$
$$\mathcal{L}_\lambda(x^*, \lambda) = 0. \tag{3.6}$$

Applying Theorem 3.2, one systematic way to search for an optimum is that we assume a particular pattern, say $x_1^* > 0$, $x_2^* = 0$, ..., $x_n^* > 0$. Then from (3.8), we get n equations: $\mathcal{L}_1(x^*, \lambda) = 0$, $x_2^* = 0$,..., $\mathcal{L}_n(x^*, \lambda) = 0$. Together with m equations in (3.6), we could

[8] Compelementary slackness holds for each component pair.

solve for the $n+m$ unknowns x^* and λ. If a solution exists, and further it satisfy the other inequality conditions required from the pattern, then it is a candidate for the optimum. There are in total 2^n such patterns to consider. Therefore, to have a complete list of candidates for the optimum, we need to repeat the above algorithm 2^n times. The simplex method for solving linear programming problems is one application of the algorithm. However, in general, this algorithm is exhaustive and exhausting. In practice, we should use our economic intuition to make good guesses about the pattern, proceed on that basis, and use second-order sufficient conditions to verify our guesses.

3.C. Inequality constraints

In this section, we introduce the inequality constraints. This is of considerable economic importance, since it is not always optimal to use up all the resources.

Suppose that the first constraint holds with inequality, that is,

$$G^1(x) \leq c_1.$$

Therefore, the problem under concern is

$$\begin{aligned} &\max_{x_1,...,x_n} F(x_1,...,x_n) \\ \text{s.t. } &G^1(x_1,...,x_n) \leq c_1, \\ &G^2(x_1,...,x_n) = c_2, ..., G^m(x_1,...,x_n) = c_m. \end{aligned}$$

Invoking the "unspent income" argument we introduced in Chapter 1, we could define a new variable x_{n+1} as follows:

$$x_{n+1} = c_1 - G^1(x). \tag{3.9}$$

Now the constraint becomes

$$G^1(x) + x_{n+1} = c_1,$$

with the additional requirement $x_{n+1} \geq 0$.

Thus, the maximization problem becomes

$$\max_{x_1,\ldots,x_n,x_{n+1}} F(x_1,\ldots,x_n)$$
$$\text{s.t. } G^1(x_1,\ldots,x_n)+x_{n+1}=c_1 \text{ and } x_{n+1}\geq 0;$$
$$G^2(x_1,\ldots,x_n)=c_2,\ldots,G^m(x_1,\ldots,x_n)=c_m.$$

We have learned how to handle such problems in Section 3.B.

Instead of transforming the problem and invoking Theorem 3.2 each time we saw such a problem, we want to find conditions for the maximization problems with the inequality constraints. Let $\widehat{\mathcal{L}}$ be the Lagrangian for the new problem with $G^1(x_1,\ldots,x_n)+x_{n+1}=c_1$ and $x_{n+1}\geq 0$, to distinguish from $\mathcal{L}$ of the old one with $G^1(x_1,\ldots,x_n)\leq c_1$. Then

$$\begin{aligned}
&\widehat{\mathcal{L}}(x_1,\ldots,x_n,x_{n+1},\lambda_1,\ldots,\lambda_m)\\
=&F(x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_m)+\lambda_1[c_1-G^1(x_1,\ldots,x_n)-x_{n+1}]+\sum_{i=2}^{m}\lambda_i[c_i-G^i(x_1,\ldots,x_n)]\\
=&F(x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_m)+\lambda_1[c_1-G^1(x_1,\ldots,x_n)]+\sum_{i=2}^{m}\lambda_i[c_i-G^i(x_1,\ldots,x_n)]-\lambda_1x_{n+1}\\
=&\mathcal{L}(x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_m)-\lambda_1x_{n+1}.
\end{aligned}$$

Applying Theorem 3.2, we have

$$\widehat{\mathcal{L}}_j=\mathcal{L}_j=0 \text{ for } j\neq n+1, \tag{3.10}$$
$$\widehat{\mathcal{L}}_{n+1}=-\lambda_1\leq 0, \text{ and } x_{n+1}\geq 0, \text{ with complementary slackness}, \tag{3.11}$$
$$\widehat{\mathcal{L}}_{\lambda_1}=\mathcal{L}_{\lambda_1}-x_{n+1}=0, \tag{3.12}$$
$$\widehat{\mathcal{L}}_{\lambda_i}=\mathcal{L}_{\lambda_i}=0 \text{ for } i\neq 1. \tag{3.13}$$

(3.10) and (3.13) are already expressed with respect to $\mathcal{L}$, so we only need to deal with (3.11) and (3.12).

By (3.12), $x_{n+1}=\mathcal{L}_{\lambda_1}$. Plugging into (3.11), we get

$$\lambda_1\geq 0 \text{ and } \mathcal{L}_{\lambda_1}\geq 0, \text{ with complementary slackness}. \tag{3.14}$$

Therefore, the solution could be expressed in terms of $\mathcal{L}$:

$$\mathcal{L}_j = 0, \text{ for } j = 1, ..., n, \tag{3.10}$$

$$\mathcal{L}_{\lambda_1} \geq 0 \text{ and } \lambda_1 \geq 0, \text{ with complementary slackness}, \tag{3.14}$$

$$\mathcal{L}_{\lambda_i} = 0 \text{ for } i = 2, 3, ..., m. \tag{3.13}$$

We could extend the above reasoning to allow all constraints to be inequalities.

Please take special note that to apply new result directly, the inequality constraints need to be of the form $G^i(x) \leq c_i$. Besides, the Lagrangian is of the form

$$\mathcal{L}(x, \lambda) = F(x) + \sum_{i=1}^{m} \lambda_i [c_i - G^i(x)].$$

Now the sign and position of the terms in the Lagrangian become important since the first-order condition involves the sign of λ_i, see (3.14).

If all constraints to be inequalities, then there is no reason in restricting $m < n$, since any number of inequality constriants can still leave a non-trivial range of variation for x.

The **Constraint qualification** needs to be altered. We only require the matrix formed by the binding constraints to have full rank.

The result is formally presented in Theorem 3.3 below:

Theorem 3.3 (Kuhn-Tucker Theorem)**.** *Suppose x is a n-dimensional vector, c an m-dimensional vector, F a function taking scalar values, G a function taking m-dimensional vector values, with $m < n$. Suppose x^* solves the following maximization problem:*

$$\max_x F(x)$$
$$s.t. \; G(x) \leq c \text{ and } x \geq 0,$$

and the constraint qualification holds, namely, the submatrix of $G_x(x^*)$ formed by taking those rows i for which $G^i(x^*) = c_i$ has the maximum possible rank. Define

$$\mathcal{L}(x, \lambda) = F(x) + \lambda \left[c - G(x)\right], \tag{3.4}$$

where λ is an m-dimensional row vector. Then there is a value of λ such that

$$\mathcal{L}_x(x^*, \lambda) \leq 0, \;\; x^* \geq 0, \;\; \textit{with complementary slackness}, \tag{3.8}$$

$$\mathcal{L}_\lambda(x^*, \lambda) \geq 0, \;\; \lambda \geq 0, \;\; \textit{with complementary slackness}. \tag{3.15}$$

Once again, an exhaustive procedure for finding a solution involves searching among all 2^{m+n} patterns from the $(m+n)$ complementary slackness conditions. And in practice, we should use our economic intuition to narrow down the search.

In the next section, we will illustrate how to apply the theorem.

3.D. Examples

Example 3.1: Quasi-linear Preferences. Suppose there are two goods x and y, whose quantities must be non-negative, and whose prices are $p > 0$ and $q > 0$ respectively. Consider a consumer with income I and the utility function[9]

$$U(x, y) = y + a\ln(x).$$

What is the consumer's optimal bundle (x, y)?

Solution. First, state the problem:

$$\max_{x,\,y} U(x, y) \equiv \max_{x,\,y} y + a\ln(x)$$
$$\text{s.t. } px + qy \leq I \text{ and } x \geq 0, \, y \geq 0.$$

This is a maximization problem with non-negative variables and inequality constriant. To solve this problem, we invoke Kuhn-Tucker Theorem.

i Form Lagrangian:

$$\mathcal{L}(x, y, \lambda) = y + a\ln(x) + \lambda\left[I - px - qy\right].$$

[9]Such preferences are called quasi-linear, because the utility function is linear in the quantity of one good.

ii First-order conditions:

$$\partial\mathcal{L}(x,y,\lambda)/\partial x = a/x - \lambda p \leq 0 \text{ and } x \geq 0, \text{ with complementary slackness;} \tag{3.16}$$

$$\partial\mathcal{L}(x,y,\lambda)/\partial y = 1 - \lambda q \leq 0 \text{ and } y \geq 0, \text{ with complementary slackness;} \tag{3.17}$$

$$\partial\mathcal{L}(x,y,\lambda)/\partial \lambda = I - px - qy \geq 0 \text{ and } \lambda \geq 0, \text{with complementary slackness.} \tag{3.18}$$

Note that given the three complementary slackness conditions, there are $2^3 = 8$ patterns to consider.

First, note that the budget constraint cannot be slack. We have discussed the intuition in Chapter 1: any income left could have been spent to increase the utility.

In terms of mathematics, suppose the budget constraint is slack, then (3.18) gives $\lambda = 0$. Plugging into (3.17) gives $1 \leq 0$, which is impossible.

We have now established that (3.18) reduces to

$$I - px - qy = 0 \text{ and } \lambda > 0. \tag{3.18'}$$

This reduces patterns to 4. Among them, notice that $x = 0$ and $y = 0$ cannot hold simultaneously, since (3.18') would be violated if $x = y = 0$.

Case I: $x = 0$ **and** $y = I/q > 0$**.** Then by (3.17), $1 - \lambda q = 0 \implies \lambda = 1/q$. By (3.16), we need $a/x - \lambda p \leq 0$. However, $a/x - p/q \leq 0 \implies p/q \geq \infty$, which is not possible. Therefore, this case would not arise.[10]

Case II: $y = 0$ **and** $x = I/p > 0$**.** Then by (3.16), $a/x - \lambda p = 0 \implies \lambda = a/I$. By (3.17), we need $1 - \lambda q \leq 0$, which is true when $I \leq aq$. This is a condition on the given parameters of the problem, and they may or may not satisfy it. If they do, the premises of the case are mutually consistent and we have a candidate for optimility. That is, if $I \leq aq$, we have a candidate solution $x^* = I/p$ and $y^* = 0$.

[10]The economic intuition is that the first small unit of x has infinite marginal utility, so it is never optimal not to consume x.

Case III: $x > 0$ and $y > 0$. Then by (3.16) and (3.17)

$$\begin{cases} a/x - \lambda p = 0 \\ 1 - \lambda q = 0 \end{cases} \Longrightarrow \begin{cases} x^* = (aq)/p \\ \lambda = 1/q \end{cases}$$

Plugging into (3.18'), we have $y^* = I/q - a$. $y^* > 0$ requires $I > aq$. Once again, the condition may or may not be satisfied; if $I > aq$, we have a candidate solution $x^* = (aq)/p$ and $y^* = I/q - a$.

To conclude, the solution is

$$\begin{cases} x^* = I/p \text{ and } y^* = 0, & \text{if } I \leq aq; \\ x^* = (aq)/p \text{ and } y^* = I/q - a, & \text{if } I > aq. \end{cases}$$

Remark. Take a closer look at the solution. When income is at a low level, all income is spent on x. However, after some point, the expenditure on x is kept constant, and all additional income is spent on y.

Example 3.2: Technological Unemployment. Suppose an economy has 300 units of labor and 450 units of land. These can be used in the production of wheat and beef. Each unit of wheat requires 2 of labor and 1 of land; each unit of beef requires 1 of labor and 2 of land.

A plan to produce x units of wheat and y units of beef is feasible if

$$2x + y \leq 300, \tag{3.19}$$

$$x + 2y \leq 450. \tag{3.20}$$

Suppose the society has an objective, or social welfare function as follows:

$$W(x, y) = \alpha \ln(x) + \beta \ln(y). \tag{3.21}$$

where $\alpha + \beta = 1$.

What is the optimal amount of wheat and beef production?

Solution. As mentioned before, it is useful to vitualize the problem. In Figure 3.1 below, we graph the constriants.

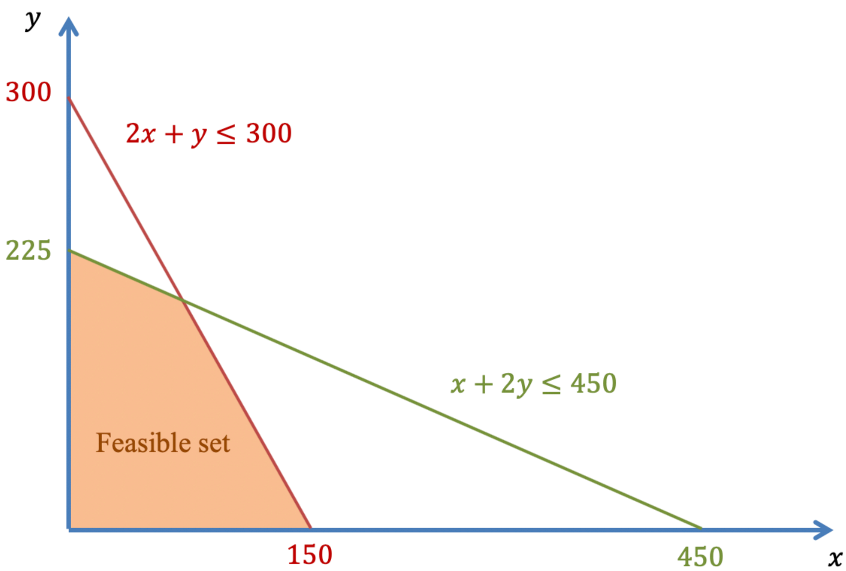

Figure 3.1: Production and unemployment

It may be tempted to think that the optimal production occurs at the intersection of the two curves. However, it may not be true.

We now apply the standard procedure to solve the problem. First, state the problem:

$$\max_{x,y} W(x,y) \equiv \max_{x,y} \alpha \ln(x) + \beta \ln(y)$$
$$\text{s.t. } 2x + y \leq 300, x + 2y \leq 450, \text{ and } x \geq 0, \, y \geq 0.$$

This is a maximization problem with non-negative variables and inequality constriant. To solve this problem, we invoke Kuhn-Tucker Theorem.

i Form Lagrangian:

$$\mathcal{L}(x,y,\lambda,\mu) = \alpha \ln(x) + \beta \ln(y) + \lambda \left[300 - 2x - y\right] + \mu \left[450 - x - 2y\right].$$

ii First-order conditions:

$$\partial\mathcal{L}/\partial x = \alpha/x - 2\lambda - \mu \le 0 \text{ and } x \ge 0, \text{ with complementary slackness;} \tag{3.22}$$

$$\partial\mathcal{L}/\partial y = \beta/y - \lambda - 2\mu \le 0 \text{ and } y \ge 0, \text{ with complementary slackness;} \tag{3.23}$$

$$\partial\mathcal{L}/\partial \lambda = 300 - 2x - y \ge 0 \text{ and } \lambda \ge 0, \text{ with complementary slackness;} \tag{3.24}$$

$$\partial\mathcal{L}/\partial \mu = 450 - x - 2y \ge 0 \text{ and } \mu \ge 0, \text{ with complementary slackness.} \tag{3.25}$$

First, we could exclude $x^* = 0$ in this example. The reasoning is the same as in the previous example. Suppose $x^* = 0$, then (3.22) becomes $\infty \le 2\lambda + \mu$. Since λ and μ are finite numbers, this cannot hold.

Similarly, we could exclude $y^* = 0$. Therefore, (3.22) and (3.23) becomes

$$\alpha/x - 2\lambda - \mu = 0 \text{ and } x > 0; \tag{3.22'}$$

$$\beta/y - \lambda - 2\mu = 0 \text{ and } y > 0. \tag{3.23'}$$

We are left with 4 patterns. Next, note that $\lambda = 0$ and $\mu = 0$ cannot hold simultaneously. If $\lambda = \mu = 0$, (3.22') and (3.23') imply $\alpha = \beta = 0$, which cannot be true since $\alpha + \beta = 1$. We are now left with 3 cases.

Case I: $\lambda = 0$ and $\mu > 0$. (3.24) and (3.25) becomes

$$300 - 2x - y \ge 0 \text{ and } \lambda = 0; \tag{3.24'}$$

$$450 - x - 2y = 0 \text{ and } \mu > 0. \tag{3.25'}$$

Plugging $\lambda = 0$ into (3.22') and (3.23'), we have

$$x = \alpha/\mu \text{ and } y = \beta/(2\mu).$$

Together with (3.25'), we could solve μ and then x^* and y^*:

$$\mu = 1/450, \text{ and } x^* = 450\alpha, \quad y^* = 225\beta.$$

It remains to check whether (3.24') holds. (3.24') requires

$$300 - 900\alpha - 225\beta \ge 0 \implies 300 - 900(\alpha + \beta) + 675\beta \ge 0 \implies \beta \ge 8/9.$$

Case II: $\lambda > 0$ and $\mu = 0$. This case could be solved following the same procedure as Case I.[11] The result is $x = 150\alpha, y = 300\beta,$ and $\beta \leq 2/3$.

Case III: $\lambda > 0$ and $\mu > 0$. (3.24) and (3.25) becomes

$$300 - 2x - y = 0 \text{ and } \lambda > 0; \tag{3.24''}$$

$$450 - x - 2y = 0 \text{ and } \mu > 0. \tag{3.25''}$$

We could solve x^* and y^* from (3.24") and (3.25"):

$$x^* = 50 \text{ and } y^* = 200.$$

Then, we need to verify (3.22') and (3.23'):

$$\begin{cases} \alpha/50 - 2\lambda - \mu = 0 \\ \beta/200 - \lambda - 2\mu = 0 \end{cases} \Longrightarrow \begin{cases} \lambda = (8 - 9\beta)/600 \\ \mu = (3\beta - 2)/300. \end{cases}$$

$\lambda > 0$ and $\mu > 0$ require

$$2/3 < \beta < 8/9.$$

To conclude, the solution is

$$\begin{cases} x^* = 150\alpha \text{ and } y^* = 300\beta, & \text{if } \beta \leq 2/3; \\ x^* = 50 \text{ and } y^* = 200, & \text{if } 2/3 < \beta < 8/9; \\ x^* = 450\alpha \text{ and } y^* = 225\beta, & \text{if } \beta \geq 8/9. \end{cases}$$

Remark. Note that the solution is at the intersection of the two constraint lines only when $2/3 < \beta < 8/9$.

[11] You should work out the case on your own.