Dynamic Optimization

Assignment 4

Due date: May 21, 2022

Submission method: QQ group

There are 4 questions in total. Please present your work in a neat and organized manner.

Question 1: Economic Development with Subsistence Consider the following social planner's problem:

$$\max_{\substack{\{c_t\}_{t=0}^{\infty} \\ \{k_{t+1}\}_{t=0}^{\infty}}} \sum_{t=0}^{\infty} \beta^t \ln(c_t - \bar{c})$$

$$\text{s.t. } c_t + k_{t+1} = k_t^{\alpha} + (1-\delta) k_t$$

with $k_0 > 0$ given. The utility function includes a constant $\bar{c} > 0$, called the subsistence consumption need.

1. Write down the (present value) Hamiltonian, the first order conditions and the transversality condition.
2. Derive the Euler equation of optimal consumption path and explain.

Question 2: Growth with Human Capital Suppose there is an education sector which is producing human capital:

$$\dot{h}(t) = B(1-u(t))\,h(t),$$

where $1-u(t)$ is the fraction of total available time at period t to be used in education. Consumption can be produced with human capital (and the fraction of time $u(t)$):

$$c(t) = u(t)\,h(t).$$

Then the social planner's problem is

$$\max_{u(t),h(t)} \int_0^\infty e^{-\rho t} \frac{(u(t)\, h(t))^{1-\sigma}}{1-\sigma} dt$$
$$\text{s.t. } \dot{h}(t) = B(1-u(t))\, h(t)$$

$h(0) > 0$ is given. $\rho > 0$ is the discount rate. B is a parameter. Assume $B > \rho$ and $(1-\sigma) B < \rho$.

1. Write down the *present* value Hamiltonian, the first order conditions and the transversality condition.
2. Write down the *current* value Hamiltonian, the first order conditions and the transversality condition.
3. Suppose u is constant over time and independent of h. So h grows at constant rate.
   a) Derive $\dot{h}/h$ in terms of the exogenous parameters σ, B, and ρ.
   b) Solve the constant allocation of time u.

Question 3: Exercise 10.2 (Optimal Growth) Consider the optimal growth problem in Example 10.2.

1. It is more convenient to work with $\phi \equiv \pi e^{\rho t}$. Interpret this variable.
2. Show that k and ϕ satisfy the pair of differential equations

$$\dot{k} = F(k) - \delta k - G(\phi)$$
$$\dot{\phi} = -\phi(F'(k) - \rho - \delta)$$

   where G is the function inverse of U'.
3. Draw the phase diagram (which should look just like Figure 10.1 but reflected upside down).

Question 4: Exercise 10.3 (Entry-Deterrence)

The demand curve in an industry at time t is given by

$$q(t) = a - bp(t),$$

where a and b are positive constants, $p(t)$ and $q(t)$ are respectively the price and the quantity. There is one large firm that sets the price, and a fringe of small firms that accept this price and sell their entire output. New fringe firms enter if the large firm charges a price greater than p^*. Write $x(t)$ for the output of the fringe firms. The initial $x(0)$ is given; $x(t)$ satisfies the differential equation

$$\dot{x}(t) = k(p(t) - p^*).$$

The large firm's sales are $q(t) - x(t)$, and its average cost is constant and equal to c. Therefore the discounted present value of its profits is

$$\int_0^\infty (p(t) - c)(a - x(t) - bp(t))e^{-\rho t}\mathrm{d}t$$

where ρ is the discount rate. Assume $p^* > c$.

1. Apply the Maximum Principle to this problem, taking x as the state variable and p as the control variable.
2. Construct the phase diagram in (x, p) space. Find the qualitative features of the optimum pricing policy of the large firm.
3. Obtain conditions on the parameters of the problem under which the competing firms retain positive sales in the limit as t goes to ∞.