Dynamic Optimization

Assignment 5

Due date: June 5, 2022

Submission method: QQ group

There are 4 questions in total. Please present your work in a neat and organized manner.

Question 1: Tree cutting problem

Basic problem A tree grows according to $k_{t+1} = k_t + 1$, where k_t is the size of the tree in period t. Assume $k_0 = 0$. In period t, the owner can cut the tree and sell the wood to obtain k_t, or he could wait for the tree to grow and cut later. Let $\beta \in (0, 1)$ be the discount factor.

1. Write down the Bellman equation.
2. How many periods will the owner wait before cutting a newly-planted (size 0) tree?

Growing new trees. Now suppose that after cutting a tree, the space occupied by the tree can immediately grow a new tree with size 0. There is a small cost $c > 0$ to plant a new tree. You can think of it as the cost of the seed. Assume that c is small such that it is efficient to grow new trees.

- Write down the Bellman equation.
- How many periods will the owner wait before cutting a newly-planted (size 0) tree?

Question 2: Life-cycle saving problem Consider the simplified life-cycle saving problem we discussed in Chapter 11. Now let the utility function be $u(c) = \sqrt{c}$. Other than the utility function, the setup is identical to the original problem.

Time is discrete. The decision is on how much of the income to spend on consumption in each period. Let $c_t \geq 0$ be the consumption in period t and k_{t+1} be the accumulated savings or debts at the beginning of period $t+1$. The budget constraint in period t is

$$c_t + k_{t+1} = k_t. \tag{1}$$

$k_0 > 0$ is given. Let $\beta \in (0,1)$ be the discount factor.

Finite-horizon Let the last period be T. We further impose the condition on terminal stock: $k_{T+1} = 0$. Apply dynamic programming to solve the model with $T = 2$.

1. Write down the Bellman equation.
2. Obtain the optimal c_0, c_1, c_2 and k_1, k_2.
3. Obtain the value function at $t = 0$: $V_0(k_0)$.

Infinite-horizon

1. Write down the Bellman equation.
2. Use the "guess and verify" method to solve the problem. Guess the value function $V(k)$.
   a) Obtain the value function.
   b) Obtain the policy function $k_{t+1} = g(k_t)$.
3. Instead of guessing the value function, guess the policy function.
   a) Obtain the policy function.
   b) Obtain the value function.
4. Try the iteration method. Start with the initial guess $V_0(k) = 0$.

Question 3: AK model Consider the following social planner's problem:

$$\max_{\substack{\{c_t\}_{t=0}^{\infty} \\ \{k_{t+1}\}_{t=0}^{\infty}}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

$$\text{s.t. } c_t + k_{t+1} = y_t = Ak_t.$$

$k_0 > 0$ is given.

1. Write down the associated Bellman equation.

2. Use the "guess and verify" method to solve the problem. Guess the value function $V(k)$.
   a) Obtain the value function.
   b) Obtain the policy function.
3. Instead of guessing the value function, guess the policy function.
   a) Obtain the policy function.
   b) Obtain the value function.

Question 4: Job market search Consider a variant of the job market search model (Example 2 of Chapter 11). Now suppose that the worker could return to any old job he had rejected. Other than this, the setup is identical to the original problem.

There is a whole spectrum of jobs paying different wages in the economy. Denote the wage offer by w. The cumulative distribution function, the probability that a randomly selected job pays w or less, is $\Phi(w)$. The corresponding density function is $\phi(w) = \Phi'(w)$. A worker must engage in search to find out how much a particular job pays. Each period, an unemployed worker draws a wage offer w. The worker could accept this offer, reject this offer but return to any old job he had rejected, or take no offers. If the worker does not take any offer, the worker stays unemployed and waits until the next period to draw another wage offer. The worker receives unemployment compensation c for each of the unemployed period. The discount factor is β.

1. Write down the Bellman equation. [Hint: Now the state variable $\hat{w}$ is the best offer the worker had observed up to a certain period.]
2. There still exists a cutoff decision rule. Characterize the cutoff $\hat{w}^*$. Compare this cutoff with the one in Example 2.