Chapter 1. Introduction

Economics is about making the best use of scarce resources. That is, we look for the optimal decision subject to a set of constraints. This course aims to outline the mathematical structures of the maximization problems and develop the economic intuition.

As an overview for the first half of the course, an example of the maximization problem is provided in the following section.

1.A. The consumer choice model

Consider the consumer choice model illustrated in Figure 1.1 below:

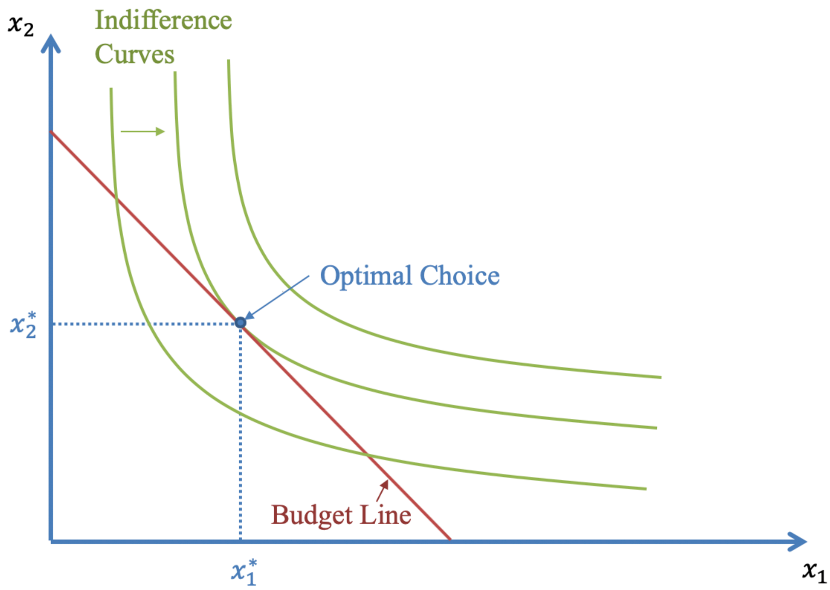

Figure 1.1: Consumer's Optimal Choice

Here, the budget line defines the consumer's economic affordability constraint. Let p_1 and p_2 be the prices of good 1 and good 2, and let I be the conumer's income. The possible quantities of good 1 (denoted by x_1) and good 2 (denoted by x_2), are given by the affordability constraint:

$$p_1 x_1 + p_2 x_2 \leq I. \tag{1.1}$$

Figure 1.2 below shows the budget constraint.

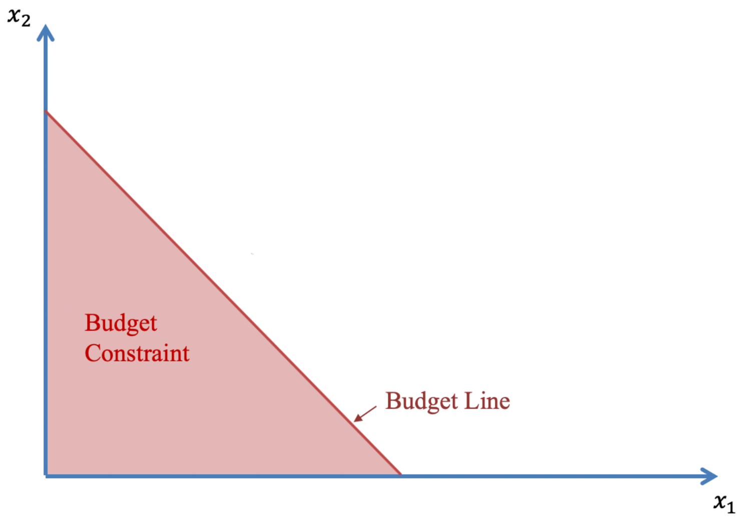

Figure 1.2: Budget Constraint

The consumer's objective function is related to her preference over x_1 and x_2. As is standard in the microeconomic theory, the consumer's preference can be represented by a utility function $U(x_1, x_2)$, which assigns each bundle (x_1, x_2) to a number, or a *utility level*, $U(x_1, x_2)$. The indifference curves in Figure 1.1 denote the bundles with the same utility level. That is, for two points (x_1', x_2') and (x_1'', x_2'') on the same indifference curve,

$$U(x_1', x_2') = U(x_1'', x_2'') = \text{ constant.}$$

Figure 1.3 below singles out the indifference curves.

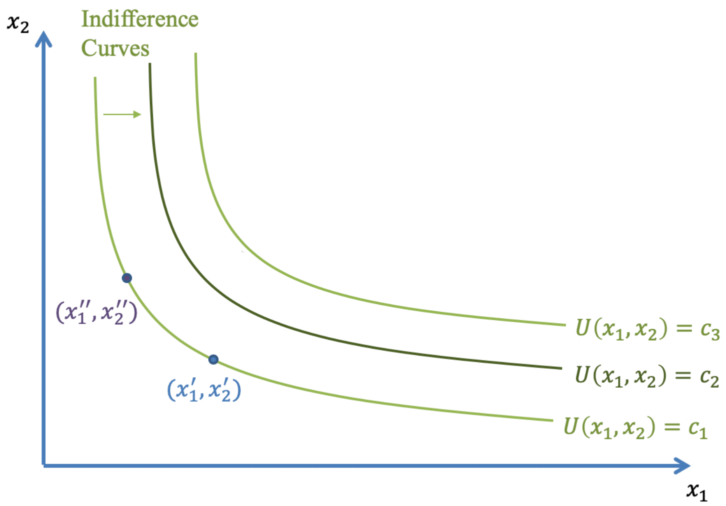

Figure 1.3: Indifference Curves

The indifference curves to the right give higher utility compared to the curves on the left. That is, in Figure 1.3,

$$c_3 > c_2 > c_1.$$

Therefore, the consumer's objective is to reach the highest indifference curve, given the budget contraint. Mathematically, the consumer's maximization problem is

$$\max_{x_1 \geq 0,\, x_2 \geq 0} U(x_1, x_2)$$
$$\text{s.t. } p_1 x_1 + p_2 x_2 \leq I.$$

In this particular example, the optimal bundle must lie on the budget line, since otherwise, any income left could have been spent to increase the utility.[1] More specifically, suppose the inequality constriant holds strict at the optimum, that is, $p_1 x_1^* + p_2 x_2^* < I$. It can be seen from the indifference curves that keeping x_2 unchanged, an increase of x_1 would increase the consumer's utility.[2] Therefore, one way to further increase the utility is to redistribute the unspent income and spend it on x_1. After the change, x_2^* is kept unchanged, whereas x_1^* increases by $\frac{I-(p_1 x_1^* + p_2 x_2^*)}{p_1}$.

Thus, based on the above observation, the constrained maximization problem could be restated as follows:

$$\max_{x_1 \geq 0,\, x_2 \geq 0} U(x_1, x_2)$$
$$\text{s.t. } p_1 x_1 + p_2 x_2 = I.$$

Graphically, the optimal should be attained where the indifference curve is tangential to the budget line.[3]

In the rest of the section, we will use verbal and geometric arguments to mathematically analyze the problem and develop the optimality condition. Two approaches are outlined here: the first one is the **arbitrage argument** and the second one is the **tangency**

[1] In Section 1.G, we will briefly discuss the case with inequality constraint.
[2] Using the terminology that would come up later, this sentence means that the marginal utility of good 1 (MU_1) is positive.
[3] See Figure 1.1.

condition using calculus. The first approach is more intuitive whereas the second one is more commonly used.

1.B. The arbitrage argument

We apply the arbitrage argument. The idea of the arbitrage argument is as follows:

(i) Start at any point, or *trial allocation*, on the budget line.

(ii) Consider a change of the bundle along the budget line. If the new bundle constitutes a higher utility, use the new bundle as the new trial allocation, and repeat Step (i) and (ii).

(iii) Stop once a better new bundle could not be found. The last bundle is the optimal bundle.

The *impossibility of finding an improvement* is served as the test of optimality.

The above process is illustrated in Figure 1.4 below.

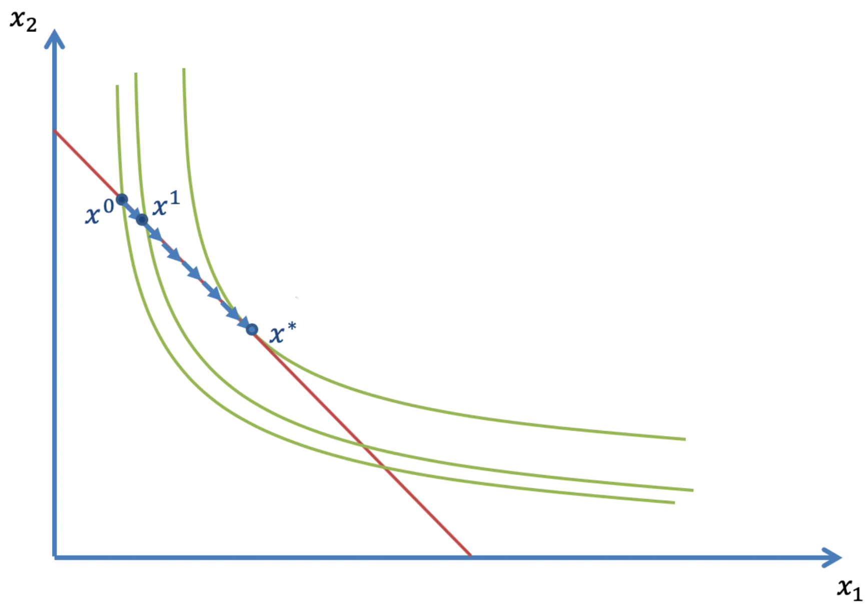

Figure 1.4: Arbitrage Argument

Now, we will investigate why this argument works, and mathematically develop the condition for the optimum.

The "arbitrage argument" and the "no-arbitrage condition". The changes we consider in the above algorithm does not entail additional expanditure, since all the points we consider are on the budget line. The changes only entail reallocation of the money from one good to the other. If the initial bundle is not optimal, a change may increase consumer's utility. For instance, in Figure 1.4, when we consider a change from the initial bundle x_0 to the new bundle x_1, consumer's utility level increases. However, when the optimum is attained, no change will ever increase the utility. In Figure 1.4, this is when we reach x^*.

The term "arbitrage" comes from the financial markets. When the financial market is **not** in the equilibrium state, participants can make "arbitrage" profit at zero cost, taking advantage of the price discrepancies in different markets. In equilibrium, there would be no such arbitrage profit. Put it differently, it is the process of people taking arbitrage profits that brings about the equilibrium. This process resembles our algorithm looking for the optimal solution. Therefore, we label the reasoning the "*arbitrage argument*" and the resulting optimality condition the "*no-arbitrage condition*".

Next, we will use the "arbitrage argument" to develop the "no-arbitrage condition". In this course, we always assume that goods are perfectly divisible.[4] Given the asssumption of perfect divisibility, the changes can occur in infinitesimal amounts, or what is called *marginal adjustments*. The standard symbol for a *marginal* change in x is $\mathrm{d}x$.

Mathematically, the "arbitrage argument" is as follows: First, suppose that the initial allocation is $x_1^0 > 0$ and $x_2^0 > 0$. Then, consider a marginal reallocation of $\mathrm{d}I > 0$ from good 2 to good 1. In physical terms, it means $\mathrm{d}x_1^0 = \mathrm{d}I/p_1$ more units of good 1 and $\mathrm{d}x_2^0 = \mathrm{d}I/p_2$ less units of good 2. Let MU_1 and MU_2 denote the marginal utilities of good 1 and good 2. The change of utility induced by the change in good 1 (good 2) is $MU_1\mathrm{d}x_1^0$ ($MU_2\mathrm{d}x_2^0$).

[4] It is a good and useful approximation. Even for seemingly indivisible goods, such as cars, there are dimiensions such as quality, that allow continuous adjustment.

Then the total change in utility is[5]

$$
\begin{aligned}
MU_1 \mathrm{d}x_1^0 + MU_2 \mathrm{d}x_2^0 =& MU_1 \mathrm{d}I/p_1 + MU_2(-\mathrm{d}I/p_2) \\
=& \left(MU_1/p_1 - MU_2/p_2\right) \mathrm{d}I. \qquad (1.2)
\end{aligned}
$$

If (1.2) is positive, that is,

$$MU_1/p_1 - MU_2/p_2 > 0,$$

the consumer will carry out this reallocation and try further reallocations in the same direction.[6] On the other hand, if the initial bundle is at the optimum, (1.2) cannot be positive. This is a part of the "no-arbitrage" criterion,

$$MU_1/p_1 - MU_2/p_2 \leq 0. \qquad \text{(no-arb 1)}$$

Next, consider a reallocation in the opposite direction, i.e., from good 1 to good 2. Following similar argument, we would arrive at the second part of the "no-arbitrage" criterion,

$$MU_1/p_1 - MU_2/p_2 \geq 0. \qquad \text{(no-arb 2)}$$

We could combine the two "no-arbitrage" criteria, (no-arb 1) and (no-arb 2), to get the following "no-arbitrage" condition:

$$MU_1/p_1 = MU_2/p_2. \qquad \text{(no-arb)}$$

The economic intuition behind the "no-arbitrage" condition is that at the optimum, the consumer should be indifferent between a marginal reallocation of any one good to the other.

[5] More rigorously, since the changes are infinitestimal, the total change is approximated by the first-order linear terms in Taylor series. You will see similar arguments relating to Taylor series in the later chapters.

[6] You can think of the process in Figure 1.4: Starting from an arbitrary x^0, each time, we consider an infinitestimal change of x along the budget line. Here, we first consider the reallocation from x_2^0 to x_1^0, that is, we consider a marginal movement to the bottom-right direction along the curve. And we check whether the change of utility is positive, i.e., $MU_1 \mathrm{d}x_1^0 - MU_2 \mathrm{d}x_2^0 > 0$.

1.C. The tangency condition using calculus

It is also possible to develop the same condition of optimality based on the tangency of the budget line and the indifference curve.

The budget line is

$$p_1x_1 + p_2x_2 = I \implies x_2 = (I/p_2) - (p_1/p_2)x_1.$$

And the slope is $-p_1/p_2$.

The slope of the indifference curve is the marginal rate of substitution ($MRS_{12} = \mathrm{d}x_2/\mathrm{d}x_1$) in consumption, and equals $(-MU_1/MU_2)$. This result could be understood as follows: Consider a marginal change of x along the indifference curve. Since the change is along the indifference curve, the marginal loss (gain) of $\mathrm{d}x_1$ units of good 1 is just compensated by the marginal gain (loss) of $\mathrm{d}x_2$ units of good 2, i.e.,

$$MU_1\mathrm{d}x_1 = MU_2(-\mathrm{d}x_2) \implies MRS = \mathrm{d}x_2/\mathrm{d}x_1 = -MU_1/MU_2.$$

At the optimum, the two slopes are equal, that is

$$p_1/p_2 = MU_1/MU_2.$$

It is easy to check that the condition above is equivalent to (no-arb) derived using the "arbitrage" argument.

When we apply the "arbitrage" argument and the tangency conditions in the previous two sections, we implicitly assume that there is an interior solution, that is, the optimum is attained when $x_1 > 0$ and $x_2 > 0$. The following section discusses corner solutions, that is, one of the good is not consumed at the optimum.

1.D. Corner solutions

Consider the optimum being attained at $x_2^* = 0$ and $x_1^* = I/p_1 > 0$. We apply the "arbitrage" argument. Now, the only possible direction of change is to decrease x_1, corresponding to (no-arb 2). Therefore, the condition for such a corner solution is

$$MU_1/p_1 - MU_2/p_2 \geq 0 \implies MU_1/MU_2 \geq p_1/p_2. \tag{1.3}$$

Graphically, Figure 1.6 shows the relative magnitude of MRS_{12} and p_1/p_2 for such a corner solution, corresponding to Equation (1.3) above. Figure 1.5 shows the relative magnitude of MRS_{12} and p_1/p_2 for such an interior solution and is put here for the purpose of comparison.

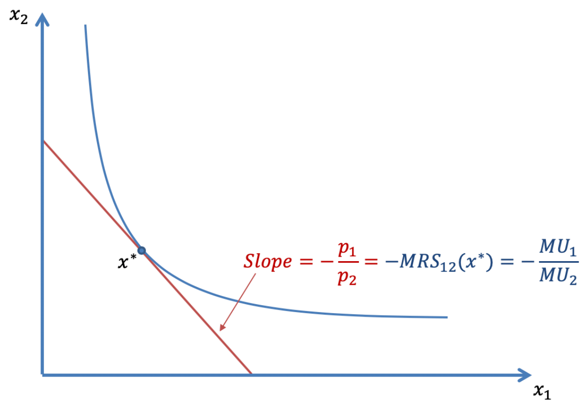

Figure 1.5: Interior solution

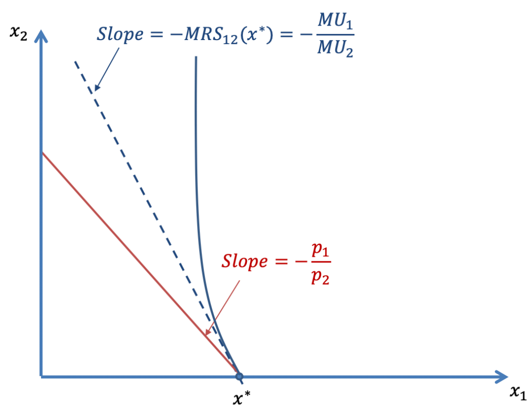

Figure 1.6: Corner solution

1.E. Marginal utility of income

In this section, we consider the marginal utility of income, that is, the marginal utility given an extra amount of dI.

Now suppose that we have an interior solution. The consumer could spend the additional income dI on good 1, buying (dI/p_1) unit of good 1, giving rise to MU_1dI/p_1 units of additional utility. Or, she could spend the addition income on good 2, which would bring MU_2dI/p_2 units of additional utility. From the "no-arbitrage" condition (no-arb), we know that the two increments are equal. Therefore, at the margin, the allocation of dI to x_1, or x_2, or even any mixture of the two, does not make any difference to the consumer. We call the utility increment per unit of maringal addition to income *the marginal utility of income*, and denote it by λ. Then dI extra units of income raise utility by λdI units.

Therefore, we have

$$\lambda = MU_1/p_1 = MU_2/p_2.$$

For the case of corner solution in Section 1.D, suppose $x_2^* = 0$ and $x_1^* = I/p_1 > 0$. Since $MU_1/p_1 \geq MU_2/p_2$, the marginal income would be spent solely on good 1 if the inequality is strict, and the consumer would be indifferent if the weak inequlity holds with equality. Therefore,

$$\lambda = MU_1/p_1 \geq MU_2/p_2.$$

1.F. Many goods and constraints

It is possible to generalize our previous consumer choice model of two goods to n goods. Let the prices be $(p_1, p_2, ..., p_n)$ and quantities be $(x_1, x_2, ..., x_n)$.
Extending the "arbitrage" argument, we must have

(i) For $x_i^* > 0$, the equality $MU_i/p_i = \lambda$ holds.

(ii) For $x_i^* = 0$, the weak inequality $MU_i/p_i \leq \lambda$ holds.

Or,

$$MU_i - \lambda p_i \begin{cases} = 0 & \text{if } x_i^* > 0; \\ \leq 0 & \text{if } x_i^* = 0. \end{cases} \tag{1.4}$$

The above alternative representation could be extended to allow several constraints. We need a separate λ for each constraint, and it can be interpreted as the marginal utility of relaxing that constraint. Details will be discussed in Chapter 3.

1.G. Non-binding Constraints

In our previous consumer choice model, the consumer benefits by spending all her income, and the budget constraint always hold with equality.
In some other applications, the constraint may not hold with equality. To illustrate the idea of the inequality constraint, we consider an extension of the consumer choice model

(even though it is not realistic here). Consider the imaginery case where the income is so large and the consumer may fail to spend it all. The budget constraint restores to (1.1) with inequality.

To solve the problem, we introduce a new good x_3, "unspent income", with $p_3 = 1$ and x_3 yielding no utility. The maximization problem becomes

$$\max_{x_1 \geq 0,\, x_2 \geq 0,\, x_3 \geq 0} U(x_1, x_2)$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 + x_3 = I.$$

Note that we have $MU_3 = 0$. That is, if $x_3^* > 0$, we must have $\lambda = MU_3 = 0$. The intuition of $\lambda = 0$ is as follows: if $x_3 > 0$, that is, the consumer does not spend all her income, it must be that the marginal utility of income is 0. $\lambda = 0$ also implies $MU_1 = MU_2 = 0$. That is, good 1 and good 2 are consumed to the point of *satiation*.

1.H. Conclusion

This chapter serves as an introduction to the theory of optimization subject to constraints. We will discuss the general theory in great detail in the chapters that follow. You will find that the conditions layed out in Section 1.F show up as Kuhn-Tucker Theorem, and the extension to the *satiated* consumer in Section 1.G appears as the principal of *Complementary Slackness*. The terminologies would become clear later.