Chapter 1. Introduction

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Introduction

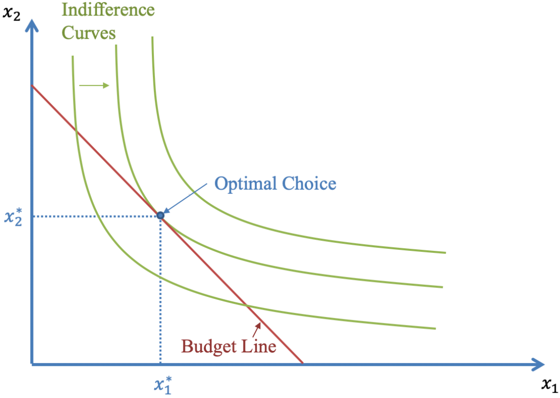
- Economics: making the best use of scarce resources.
- That is, we look for the optimal decision subject to a set of constraints.
- This course aims to outline the mathematical structures of the maximization problems and develop the economic intuition.

Introduction

As an overview for the first half of the course, an example of the maximization problem is provided in the following section.

1.A. The consumer choice model

Consider the consumer choice model illustrated below:



Budget Constraint/Budget Line

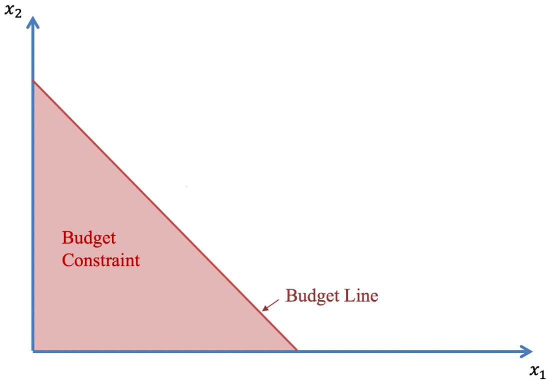
- Let p_1 and p_2 be the prices of good 1 and good 2, x_1 and x_2 be the quantities of good 1 and good 2, and I be the conumer's income.

- The possible quantities are given by the affordability constraint, called *Budget Constraint*:

$$p_1 x_1 + p_2 x_2 \leq I. \tag{1.1}$$

Budget Constraint/Budget Line

Figure below shows the budget constraint.



Budget Line: $p_1x_1 + p_2x_2 = I \implies x_2 = -\frac{p_1}{p_2}x_1 + \frac{I}{p_2}$

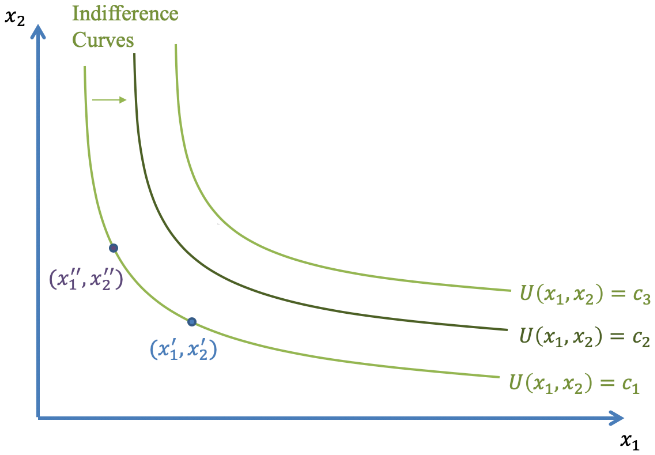
Indifference Curves

- Consumer's preference can be represented by a utility function $U(x_1, x_2)$.
- The indifference curves denote the bundles with the same utility level.
- That is, for two points (x_1', x_2') and (x_1'', x_2'') on the same indifference curve,

$$U(x_1', x_2') = U(x_1'', x_2'') = \text{ constant.}$$

Consumer's Objective Function

Figure below singles out the indifference curves.



Maximization Problem

- Consumer's objective is to reach the highest indifference curve, given the budget contraint.

- Mathematically, Consumer's maximization problem is

$$\max_{x_1 \geq 0,\, x_2 \geq 0} U(x_1, x_2)$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 \leq I.$$

Maximization Problem

- In this particular example, the optimal bundle must lie on the budget line, since otherwise, any income left could have been spent to increase the utility.

- More specifically, suppose $p_1x_1^* + p_2x_2^* < I$.

- One way to increase utility is to keep x_2^* unchanged and increase x_1^*.

- Therefore, $p_1x_1^* + p_2x_2^* = I$.

Maximization Problem

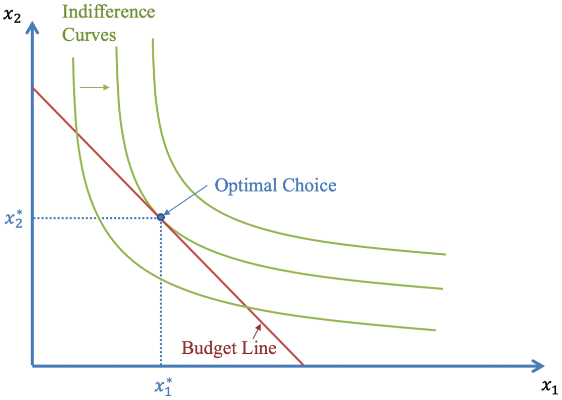
Based on the above observation, the constrained maximization problem could be restated as follows:

$$\max_{x_1 \geq 0,\, x_2 \geq 0} U(x_1, x_2)$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 = I.$$

Maximization Problem

Graphically, the optimal should be attained where the indifference curve is tangential to the budget line.



Maximization Problem

- In the rest of the section, we will use verbal and geometric arguments to mathematically analyze the problem and develop the **optimality condition**.

- Two approaches are outlined here: the first one is the **arbitrage argument** and the second one is the **tangency condition using calculus**.

- The first approach is more intuitive whereas the second one is more commonly used.

1.B. The arbitrage argument

The idea of the arbitrage argument is as follows:

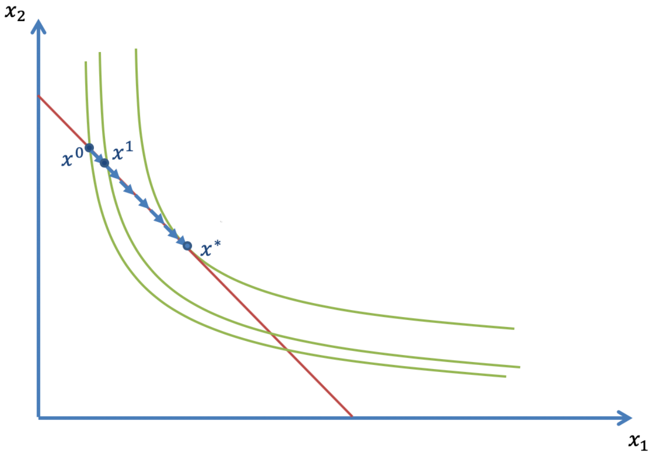
(i) Start at any point, or *trial allocation*, on budget line.

(ii) Consider a change of the bundle along the budget line. If the new bundle constitutes a higher utility, use the new bundle as the new trial allocation, and repeat Step (i) and (ii).

(iii) Stop once a better new bundle could not be found. The last bundle is the optimal bundle.

The arbitrage argument

The above process is illustrated in Figure below.



The arbitrage argument

The *impossibility of finding an improvement* is served as the test of optimality.

The arbitrage argument

Now, we will investigate why this argument works, and mathematically develop the condition for the optimum.

The arbitrage argument

- The changes we consider in the above algorithm does not entail additional expenditure, since all the points we consider are on the budget line.

- The changes only entail reallocation of the money from one good to the other.

- If the initial bundle is not optimal, a change may increase consumer's utility.

“arbitrage”

- The term “arbitrage” comes from the financial markets.
- When the financial market is **not** in the equilibrium state, participants can make “arbitrage” profit at zero cost, taking advantage of the price discrepancies in different markets.
- In equilibrium, there would be no such arbitrage profit.

“arbitrage”

- Put it differently, it is the process of people taking arbitrage profits that brings about the equilibrium.

- This process resembles our algorithm looking for the optimal solution.

- Therefore, we label the reasoning the “*arbitrage argument*” and the resulting optimality condition the “*no-arbitrage condition*”.

The "arbitrage argument" and "no-arbitrage condition".

- Next, we will use the "arbitrage argument" to develop the "no-arbitrage condition".

- In this course, we always assume that goods are perfectly divisible.

- Given the asssumption of perfect divisibility, the changes can occur in infinitesimal amounts, or what is called *marginal adjustments*. The standard symbol for a *marginal* change in x is $\mathrm{d}x$.

The "arbitrage argument" and "no-arbitrage condition".

Mathematically, the "arbitrage argument" is as follows:

- First, suppose that the initial allocation is $x_1^0 > 0$ and $x_2^0 > 0$.
- Then, consider a marginal reallocation of $\mathrm{d}I > 0$ from good 2 to good 1.
- In physical terms, it means $\mathrm{d}x_1^0 = \mathrm{d}I/p_1$ more units of good 1 and $\mathrm{d}x_2^0 = \mathrm{d}I/p_2$ less units of good 2.

The "arbitrage argument" and "no-arbitrage condition".

- Let MU_1 and MU_2 denote the marginal utilities of good 1 and good 2.

- The change of utility induced by the change in good 1 (good 2) is $MU_1 \mathrm{d}x_1^0$ ($MU_2 \mathrm{d}x_2^0$).[1]

- Then the total change in utility is

$$
\begin{aligned}
MU_1 \mathrm{d}x_1^0 + MU_2 \mathrm{d}x_2^0 =& MU_1 \mathrm{d}I/p_1 + MU_2(-\mathrm{d}I/p_2) \\
=& \left(MU_1/p_1 - MU_2/p_2\right) \mathrm{d}I. \quad (1.2)
\end{aligned}
$$

[1] first-order approximation

The "arbitrage argument" and "no-arbitrage condition".

- If (1.2) is positive, that is, $MU_1/p_1 - MU_2/p_2 > 0$, the consumer will carry out this reallocation.
- On the other hand, if initial bundle is optimal, (1.2) cannot be positive.
- This is a part of the "no-arbitrage" criterion,

$$MU_1/p_1 - MU_2/p_2 \leq 0. \qquad \text{(no-arb 1)}$$

The “arbitrage argument” and “no-arbitrage condition”.

- Next, consider a reallocation in the opposite direction, i.e., from good 1 to good 2.

- Following similar argument, we would arrive at the second part of the “no-arbitrage” criterion,

$$MU_1/p_1 - MU_2/p_2 \geq 0. \qquad \text{(no-arb 2)}$$

The “arbitrage argument” and “no-arbitrage condition”.

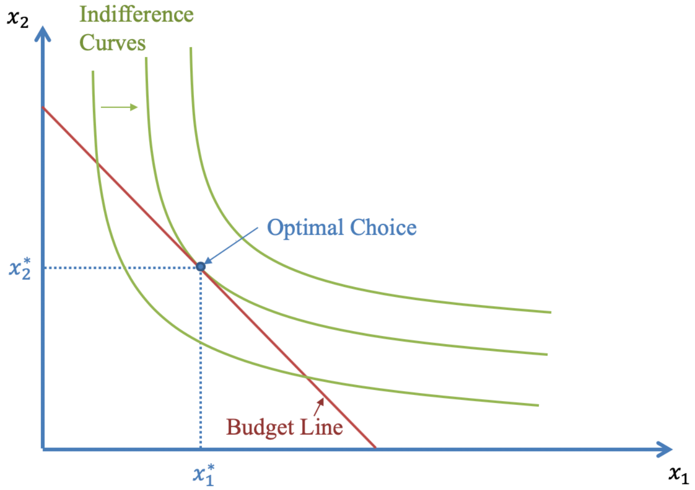
We could combine the two “no-arbitrage” criteria, (no-arb 1) and (no-arb 2), to get the following “no-arbitrage” condition:

$$MU_1/p_1 = MU_2/p_2. \tag{no-arb}$$

The “arbitrage argument” and “no-arbitrage condition”.

The economic intuition behind the “no-arbitrage” condition is that at the optimum, the consumer should be indifferent between a marginal reallocation of any one good to the other.

1.C. The tangency condition using calculus



The tangency condition using calculus

The budget line is

$$p_1 x_1 + p_2 x_2 = I \implies x_2 = -(p_1/p_2)x_1 + (I/p_2).$$

And the slope is $-p_1/p_2$.

The tangency condition using calculus

The slope of the indifference curve is the marginal rate of substitution ($MRS_{12} = \mathrm{d}x_2/\mathrm{d}x_1$), and equals $(-MU_1/MU_2)$.

- Consider a marginal change of x along indifference curve.
- Since change is along indifference curve, the marginal loss (gain) of $\mathrm{d}x_1$ units of good 1 is just compensated by the marginal gain (loss) of $\mathrm{d}x_2$ units of good 2, i.e.,

$$MU_1 \mathrm{d}x_1 = MU_2(-\mathrm{d}x_2) \implies MRS = \frac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\frac{MU_1}{MU_2}.$$

The tangency condition using calculus

At the optimum, the two slopes are equal, that is

$$p_1/p_2 = MU_1/MU_2.$$

It is easy to check that the condition above is equivalent to (no-arb) derived using the “arbitrage” argument.

1.D. Corner Solutions

- When we apply the “arbitrage” argument and the tangency conditions in the previous two sections, we implicitly assume that there is an interior solution, that is, the optimum is attained when $x_1 > 0$ and $x_2 > 0$.

- The following section discusses corner solutions, that is, one of the good is not consumed at the optimum.

Corner Solutions

- Consider optimum attained at $x_2^* = 0$ & $x_1^* = I/p_1 > 0$.

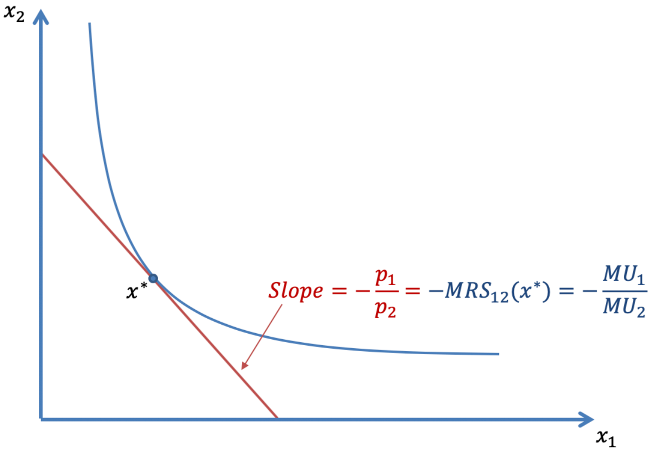
- We apply the "arbitrage" argument.

- Now, the only possible direction of change is to decrease x_1, corresponding to (no-arb 2).

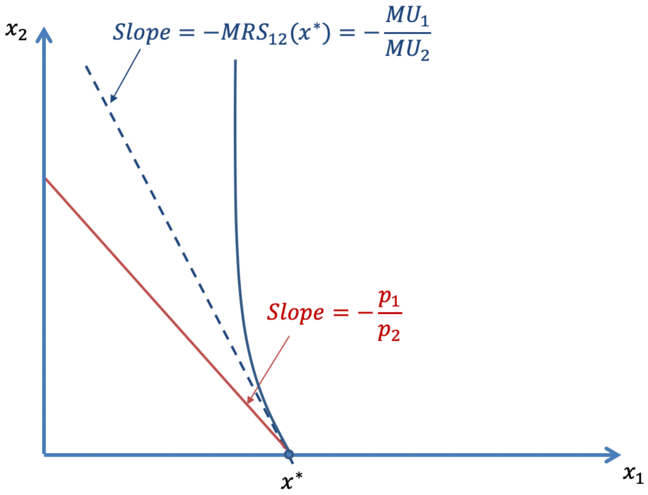
- Therefore, the condition for such a corner solution is

$$MU_1/MU_2 \geq p_1/p_2. \tag{1.3}$$

Interior Solution



Corner Solution



1.E. Marginal utility of income

In this section, we consider the marginal utility of income, that is, the marginal utility given an extra amount of $\mathrm{d}I$.

Marginal utility of income

- Now suppose that we have an **interior solution**.
- The consumer could spend the additional income $\mathrm{d}I$ on good 1, buying $(\mathrm{d}I/p_1)$ unit of good 1, giving rise to $MU_1\mathrm{d}I/p_1$ units of additional utility.
- Or, she could spend the addition income on good 2, which would bring $MU_2\mathrm{d}I/p_2$ units of additional utility.

Marginal utility of income

- From the "no-arbitrage" condition (no-arb), we know that the two increments are equal.

- Therefore, at the margin, the allocation of $\mathrm{d}I$ to x_1, or x_2, or even any mixture of the two, does not make any difference to the consumer.

Marginal utility of income

- We call the utility increment per unit of maringal addition to income *the marginal utility of income*, and denote it by λ.

- Then $\mathrm{d}I$ extra units of income raise utility by $\lambda \mathrm{d}I$ units.

- Therefore, we have

$$\lambda = MU_1/p_1 = MU_2/p_2.$$

Marginal utility of income

- Now suppose that we have a **corner solution**.
- Suppose $x_2^* = 0$ and $x_1^* = I/p_1 > 0$.
- Since $MU_1/p_1 \geq MU_2/p_2$, the marginal income would be spent solely on good 1 if the inequality is strict, and the consumer would be indifferent if the weak inequlity holds with equality.
- Therefore, $\lambda = MU_1/p_1 \geq MU_2/p_2$.

1.F. Many goods and constraints

It is possible to generalize our previous consumer choice model of two goods to n goods. Let the prices be $(p_1, p_2, ..., p_n)$ and quantities be $(x_1, x_2, ..., x_n)$.

Many goods

Recall no "arbitrage" argument:

- For interior solutions:

$$MU_1/p_1 = MU_2/p_2 = \lambda. \tag{no-arb}$$

- For corner solutions: if $x_2^* = 0$, then

$$\lambda = MU_1/p_1 \geq MU_2/p_2.$$

Many goods

Extending the "arbitrage" argument, we must have

(i) For $x_i^* > 0$, the equality $MU_i/p_i = \lambda$ holds.

(ii) For $x_i^* = 0$, the weak inequality $MU_i/p_i \leq \lambda$ holds.

Or,

$$MU_i - \lambda p_i \begin{cases} = 0 & \text{if } x_i^* > 0; \\ \leq 0 & \text{if } x_i^* = 0. \end{cases} \tag{1.4}$$

Many constraints

- The condition

$$MU_i - \lambda p_i \begin{cases} = 0 & \text{if } x_i^* > 0; \\ \leq 0 & \text{if } x_i^* = 0. \end{cases} \tag{1.4}$$

  could be extended to allow several constraints.

- We need a separate λ for each constraint, and it can be interpreted as the marginal utility of relaxing that constraint.

1.G. Non-binding Constraints

- In our previous consumer choice model, the consumer benefits by spending all her income, and the budget constraint always hold with equality.

- In some other applications, the constraint may not hold with equality.

Non-binding Constraints

- To illustrate the idea of the inequality constraint, we consider an extension of the consumer choice model (even though it is not realistic here).

- Consider the imaginery case where the income is so large and the consumer may fail to spend it all.

- The budget constraint restores to

$$p_1 x_1 + p_2 x_2 \leq I.$$

Non-binding Constraints

- To solve the problem, we introduce a new good x_3, "unspent income", with $p_3 = 1$ and x_3 yielding no utility.

- The maximization problem becomes

$$\max_{x_1 \geq 0,\, x_2 \geq 0,\, x_3 \geq 0} U(x_1, x_2)$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 + x_3 = I.$$

Non-binding Constraints

- Note that we have $MU_3 = 0$.

- That is, if $x_3^* > 0$, we must have $\lambda = MU_3 = 0$.

- The intuition of $\lambda = 0$ is as follows: if $x_3 > 0$, that is, the consumer does not spend all her income, it must be that the marginal utility of income is 0.

- $\lambda = 0$ also implies $MU_1 = MU_2 = 0$. That is, good 1 and good 2 are consumed to the point of *satiation*.

1.H. Conclusion

- This chapter serves as an introduction to the theory of optimization subject to constraints.

- We will discuss the general theory in great detail in the chapters that follow.

- You will find that the conditions layed out in Section 1.F show up as Kuhn-Tucker Theorem, and the extension to the *satiated* consumer in Section 1.G appears as the principal of *Complementary Slackness*.