Chapter 6. Convex Sets and Their Separations

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Introduction

- In the previous chapters, we have learned first-order necessary conditions for constrained maximization problems.
- We also mentioned that those conditions may not be sufficient.
- In this and the following two chapters, we will discuss sufficient conditions.

6.A. The Separation Property

- Consider the following maximization problem:

$$\max_{x} F(x)$$

$$\text{s.t. } G(x) \leq c,$$

  where $G(x) \leq c$ is a scalar constraint.

- x^*: the optimal choice; v^*: the maximum value.

- We are now interested to know the properties of the functions F and G that ensure the maximum.

The Separation Property

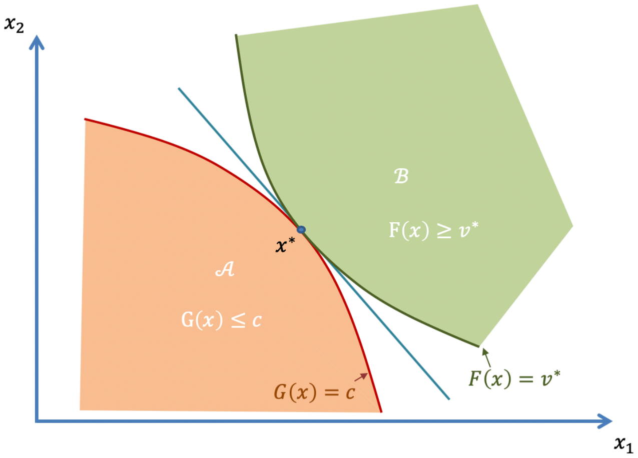


Figure 6.1: Separation by the common tangent

The Separation Property

- To get some idea about the general property, we will interpret the solution in terms of curvatures of F and G.

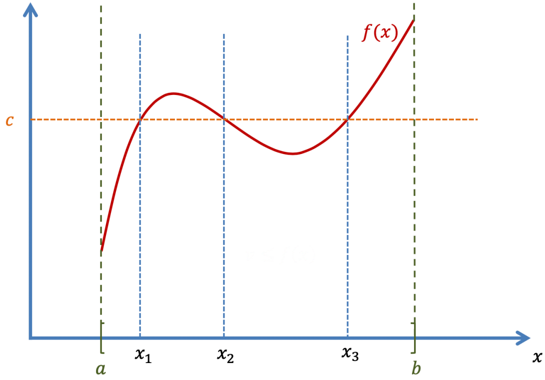
- New concepts are needed for our discussion.

Contour Sets

Definition 6.A.1 (Lower Contour Set)**.** For a function $f : \mathcal{S} \subset \mathbb{R}^N \to \mathbb{R}$, the lower contour set of f for the value $c \in \mathbb{R}$ is $\{x | f(x) \leq c\}$.

Definition 6.A.2 (Upper Contour Set)**.** For a function $f : \mathcal{S} \subset \mathbb{R}^N \to \mathbb{R}$, the upper contour set of f for the value $c \in \mathbb{R}$ is $\{x | f(x) \geq c\}$.

Contour Sets



- Lower contour set of f for the value c: $[a, x_1] \bigcup [x_2, x_3]$;
- Upper contour set of f for the value c: $[x_1, x_2] \bigcup [x_3, b]$.

The Separation Property

In Figure 6.1,

- The lower contour set of G for c is Set $\mathcal{A}$.
- The upper contour set of F for v^* is Set $\mathcal{B}$.
- Such curvatures ensure a maximum.

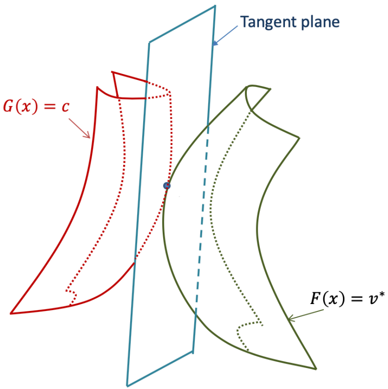
The Separation Property

Question: What is the general property of such curvatures?

- The sets $\mathcal{B}$ and $\mathcal{A}$ lie one to each side of their common tangent, with only their common point x^* on that line.
- In other words, the common tangent separates the x-plane into two halves, each containing one of the two sets.

The Separation Property

- For three-variables, the common tangent is a plane.



- In higher dimensions, it will be a hyperplane.

The Separation Property

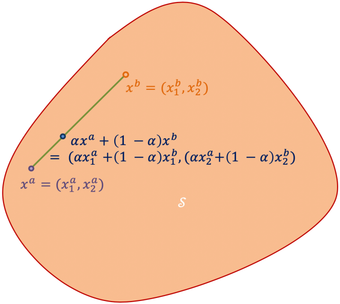
- This separation property is the crucial property that allows us to find the maxima, and obtain sufficient conditions for the maximization problem.

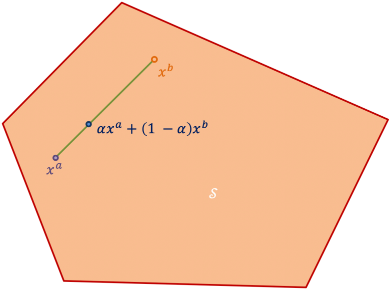
- We will next examine the explicit conditions on the functions F and G that ensure the right curvature.

6.B. Convex Sets and Functions

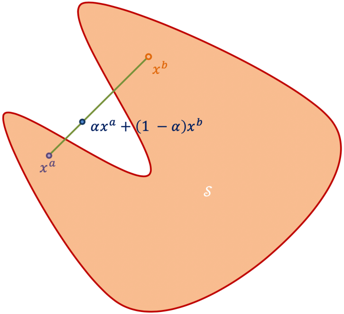
Definition 6.B.1 (Convex Set)**.** A set $\mathcal{S}$ of points in n-dimensional space is called convex if, given any two points $x^a = (x_1^a, x_2^a, ..., x_n^a)$ and $x^b = (x_1^b, x_2^b, ..., x_n^b)$ in $\mathcal{S}$ and any real number $\alpha \in [0, 1]$, the point $\alpha x^a + (1 - \alpha)x^b = (\alpha x_1^a + (1 - \alpha)x_1^b, ..., \alpha x_n^a + (1 - \alpha)x_n^b)$ is also in $\mathcal{S}$.

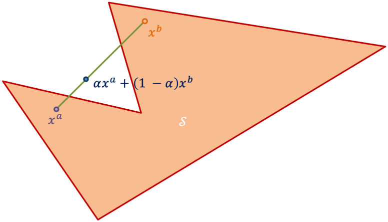
Convex Sets


$x^b = (x_1^b, x_2^b)$
$\alpha x^a + (1 - \alpha) x^b$
$= (\alpha x_1^a + (1 - \alpha) x_1^b, (\alpha x_2^a + (1 - \alpha) x_2^b)$
$x^a = (x_1^a, x_2^a)$
S


x^b
$\alpha x^a + (1 - \alpha) x^b$
x^a
S

Non-Convex Sets





Quasi-Convexity

- Apply the concept of convex sets to the lower contour set of G, we could reinterpret the bulging outward curvature as follows: the lower contour set of G is convex, or

$$\text{the set } \{x|G(x) \leq c\} \text{ is convex.} \tag{6.1}$$

- Algebraically, for all $\alpha \in [0, 1]$,

$$G(x^a) \leq c \text{ and } G(x^b) \leq c \implies G(\alpha x^a + (1-\alpha)x^b) \leq c.$$

- We need to invoke the condition for all c.

Quasi-Convexity

- The condition (6.1) with a general c is equivalent to

$$G(\alpha x^a + (1-\alpha)x^b) \leq \max\{G(x^a), G(x^b)\}, \qquad (6.2)$$

for all x^a, x^b and for all $\alpha \in [0,1]$.

- A function G satisfying this condition is called quasi-convex.

Quasi-Convexity

Definition 6.B.2 (Quasi-convex Function)**.** A function $f : \mathcal{S} \to \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, is quasi-convex

- if the set $\{x | f(x) \leq c\}$ is convex for all $c \in \mathbb{R}$,
- or equivalently, if

$$f(\alpha x^a + (1 - \alpha)x^b) \leq \max\{f(x^a), f(x^b)\}, \tag{6.3}$$

for all x^a, x^b and for all $\alpha \in [0, 1]$.

Quasi-Convexity

Next, we show the equivalence of

(a) The set $\{x|f(x) \leq c\}$ is convex for all $c \in \mathbb{R}$;

(b) $f(\alpha x^a + (1-\alpha)x^b) \leq \max\{f(x^a), f(x^b)\}$, for all x^a, x^b and for all $\alpha \in [0, 1]$.

Quasi-Concavity

- The parallel condition on F is that the upper contour set of F is convex, or F is quasi-concave.

Definition 6.B.3 (Quasi-concave Function)**.** A function $f : \mathcal{S} \rightarrow \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, is quasi-concave

- if the set $\{x|f(x) \geq c\}$ is convex for all $c \in \mathbb{R}$,
- or equivalently, if $f(\alpha x^a + (1-\alpha)x^b) \geq \min\{f(x^a), f(x^b)\}$, for all x^a, x^b and for all $\alpha \in [0, 1]$.

A digression: quasi-convexity and convexity

The quasi in Definition 6.B.2 and 6.B.3 serves to distringuish them from stronger properties of convexity and concavity.

Definition 6.B.4 (Convex Function)**.** A function $f : \mathcal{S} \to \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, is convex if

$$f(\alpha x^a + (1-\alpha)x^b) \leq \alpha f(x^a) + (1-\alpha) f(x^b), \qquad (6.4)$$

for all x^a, x^b and for all $\alpha \in [0,1]$.

A digression: quasi-convexity and convexity

- (6.4) convexity implies (6.3) quasi-convexity since

$$f(\alpha x^a + (1-\alpha)x^b) \underbrace{\le}_{(6.4)} \alpha f(x^a) + (1-\alpha) f(x^b)$$

$$\le \alpha \max\{f(x^a), f(x^b)\} + (1-\alpha)\max\{f(x^a), f(x^b)\}$$

$$= \max\{f(x^a), f(x^b)\}.$$

- In other words, a convex function must be quasi-convex.

A digression: quasi-concavity and textconcavity

- Similarly, we could define concavity and compare it with quasi-concavity.

Definition 6.B.5 (Concave Function). A function $f : \mathcal{S} \to \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, is concave if

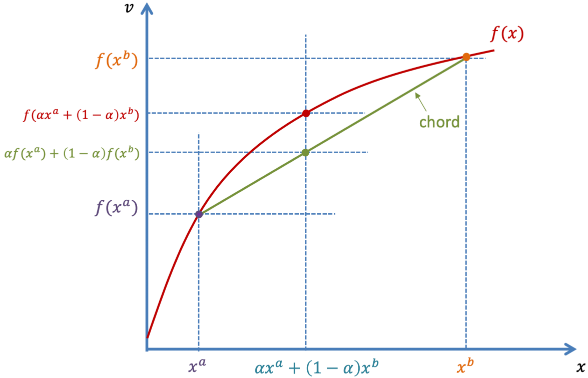
$$f(\alpha x^a + (1-\alpha)x^b) \geq \alpha f(x^a) + (1-\alpha)f(x^b), \qquad (6.5)$$

for all x^a, x^b and for all $\alpha \in [0, 1]$.

- Following the same logic, we could show that a concave function must be quasi-concave.

Concave Functions

$$f(\alpha x^a + (1-\alpha)x^b) \geq \alpha f(x^a) + (1-\alpha)f(x^b) \qquad (6.5)$$



The graph of the function lies on or above the chord joining any two points of it.

Concave Functions

- An alternative interpretation of a concave function is sometimes useful.

- Consider the $(n+1)$-dimensional space consisting of points like (x, v).

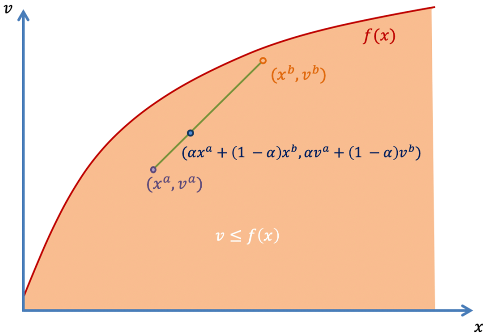
- Define the set $\mathcal{F} = \{(x, v) | v \leq f(x)\}$.

Claim. f is a concave function if and only if

$\mathcal{F}$ is a convex set.

Concave Functions

Claim. f is a concave function iff $\mathcal{F}$ is a convex set.



The claim means that the concave function f traps a convex set $\mathcal{F}$ underneath its graph.

Two More Concepts: Interior Point

Definition 6.B.6 (Interior Point)**.** A point $x^o \in \mathcal{S}$ is called an interior point if there exists a real number $r > 0$ such that for all x such that $\|x - x^o\| < r$, we have $x \in \mathcal{S}$.

- That is, a point $x^o \in \mathcal{S}$ is an interior point if all points within the distance of r from the point x^o are in $\mathcal{S}$.

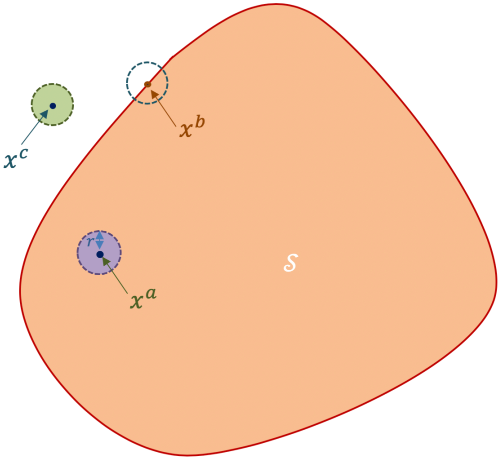
- In the plane, such points will form a disc of radius r centered at x^o.

Two More Concepts: Boundary Point

Definition 6.B.7 (Boundary Point)**.** A point $x^o \in \mathcal{S}$ is called an boundary point if for any real number $r > 0$, there exist x, y such that $\|x - x^o\| < r$, $\|y - x^o\| < r$ and $x \in \mathcal{S}$, $y \notin \mathcal{S}$.

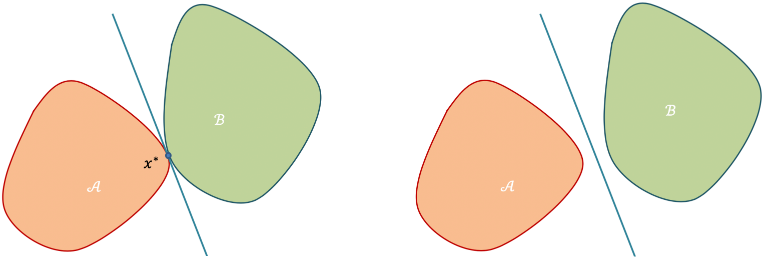
- That is, a boundary point of S is interior neither to $\mathcal{S}$ nor to the rest of the space.

Interior and Boundary Points



Separation

- Separation is possible.

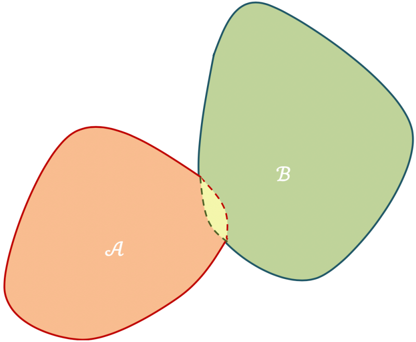


(a) common tangent

(b) no points in common

Separation

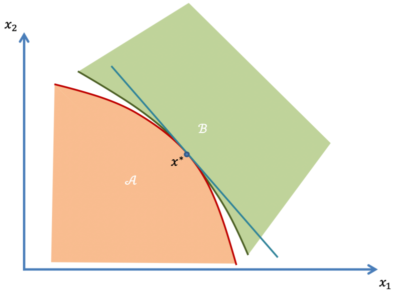
- Separation is impossible.



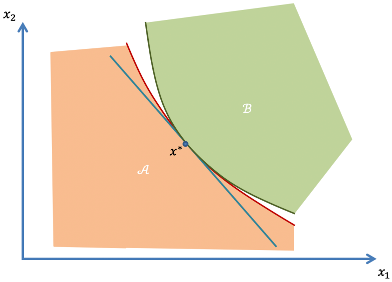
(c) interior points in common

Separation

- Convexity of the sets is important.



(a)



(b)

Separation Theorem

Theorem 6.1 (Separation Theorem)**.** If $\mathcal{B}$ and $\mathcal{A}$ are two convex sets, that have no interior points in common, and at least one of the sets has a non-empty interior, then we can find a non-zero vector p and a number b such that the hyperplane $px = b$ separates the two sets, or

$$px = \sum_{i=1}^{n} p_i x_i \begin{cases} \leq b & \text{for all } x \in \mathcal{A} \\ \geq b & \text{for all } x \in \mathcal{B}. \end{cases} \tag{6.6}$$

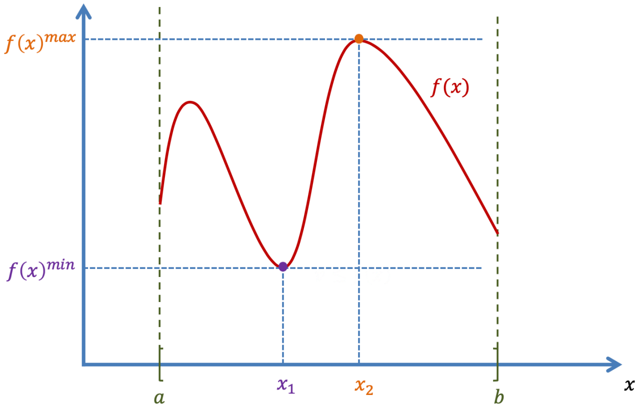
6.C. Optimization by Separation

Existence of Solution

In most economic applications, the functions F and G are well-behaved, and the existence of solution is ensured by the Extreme Value Theorem.

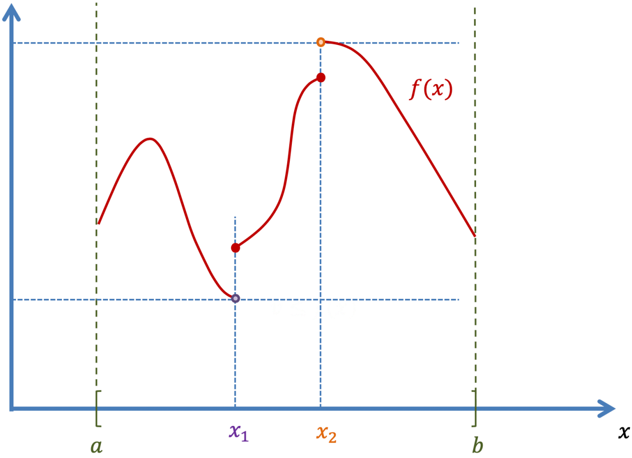
Theorem (Extreme Value Theorem)**.** If f is a continuous function defined on a closed and bounded set $\mathcal{A} \subset \mathbb{R}^N$, then f attains an absolute maximum and absolute minimum value on $\mathcal{A}$.

Extreme Value Theorem



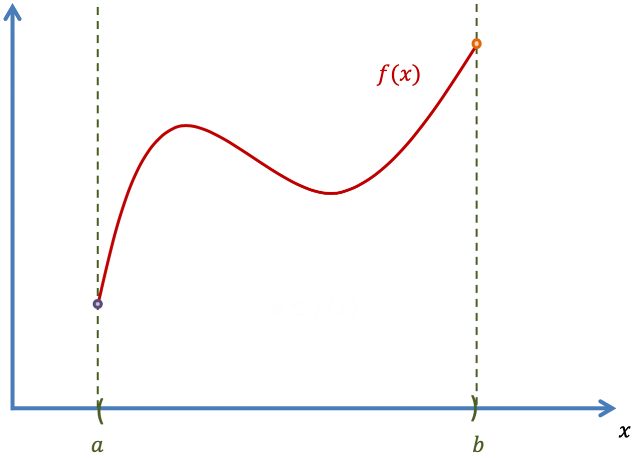
all conditions met

Extreme Value Theorem



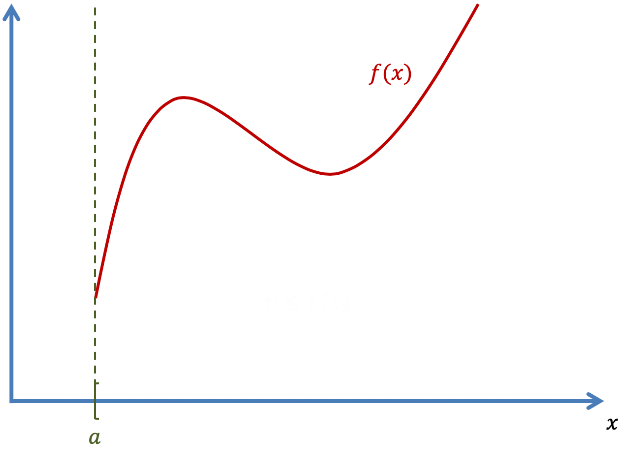
(b) not continuous

Extreme Value Theorem



(c) not closed

Extreme Value Theorem



(d) not bounded

Existence of Solution

- For our discussion in this chapter, we impose F and G being continuous, and the constraint set being bounded.
- Once we impose thses conditions, we could apply Extreme Value Theorem and existence of maximum is ensured.
- Again, for our applications, these conditions are almost always satisfied and the existence of an optimum is usually not a problem.

Optimization by Separation

Besides the above conditions, we also impose

- F quasi-concave and
- G quasi-convex,

so that all conditions assumed in Figure 6.1 are met and the Seperation Theorem (Theorem 6.1) applies.

Optimization by Separation

- The equation of separating common tangent: $px = b$.
- The equation is unaffected if we multiply it through by -1, but will reverse directions of inequalities in

$$px = \sum_{i=1}^{n} p_i x_i = \begin{cases} \leq b & \text{for all } x \in \mathcal{A} \\ \geq b & \text{for all } x \in \mathcal{B}. \end{cases} \tag{6.6}$$

- To ensure that the inequalities are consistent for the set $\mathcal{B}$ and $\mathcal{A}$ in Figure 6.1 and in Theorem 6.1, we choose $p_1, p_2 > 0$.

Optimization by Separation

- Since x^* lies on the separating tangent, so $px^* = b$.
- Therefore,

$$px = \sum_{i=1}^{n} p_i x_i = \begin{cases} \leq b & \text{for all } x \in \mathcal{A} \\ \geq b & \text{for all } x \in \mathcal{B}. \end{cases} \tag{6.6}$$

tells us that x^* gives the largest value of px among all points in $\mathcal{A}$, that is, among all points satisfying $G(x) \leq c$.

- Similarly, x^* gives the smallest value of px among all points in $\mathcal{B}$, that is, among all points satisfying $F(x) \geq v^*$.

Optimization by Separation

Theorem 6.2 (Optimization by Separation)**.** Given a quasi-concave function F and a quasi-convex function G, the point x^* maximizes $F(x)$ subject to $G(x) \leq c$ if, and only if, there is a non-zero vector p such that

(i) x^* maximizes px subject to $G(x) \leq c$, and

(ii) x^* miminizes px subject to $F(x) \geq v^*$.

Optimization by Separation

- Generalization to several constraints is straightforward.
- Set $\mathcal{A}_i$ of points for which $G^i(x) \leq c_i$ is convex if G^i is quasi-convex.
- If this is so for all i, then set $\mathcal{A}$ of points satisfying all constraints, being intersection of the convex sets $\mathcal{A}_i$, is also convex.
- Then Theorem 6.2 applies.

Optimization by Separation

- Note that Theorem 6.2 provides an "if and only if" result.

- That is, the conditions are both necessary and sufficient for optimality.

- But the problem with this theorem is that the conditions are not easy to verify in practical applications.

- In the next two chapters, we shall see sufficient conditions that are more useful in this regard.

Decentralization

- The real benefit from splitting the maximization problem into two separate problems comes from its economic interpretation.

- It raises the possibility of decentralizing optimal resource allocations using prices.

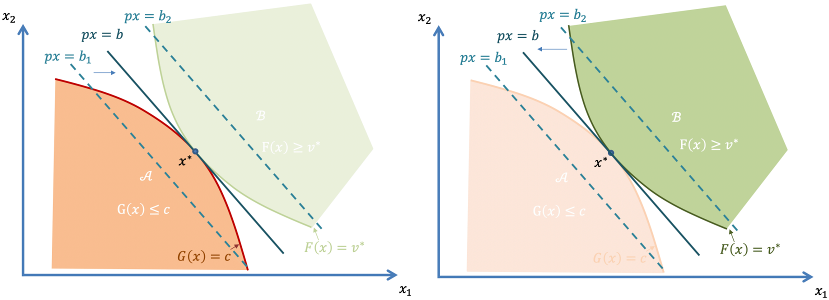
Decentralization

- Consider x as the production-cum-consumption vector, the constraints reflect limited resource availability, and the objective is the utility function.

- Now interpret p as the row vector of prices of outputs.

- The original problem of social optimization (Figure 6.1) can be decentralized.

Decentralization



(a) Producer Problem (b) Consumer Problem

Remark: Advantages of Separation

This separation of decision has two advantages:

i. Informational: producer does not need to know consumer's taste; and consumer does not need to know production technology.

ii. Incentives: the process relies on self-interest of each side to ensure effective implementation of optimum.

Remark: Relative Prices

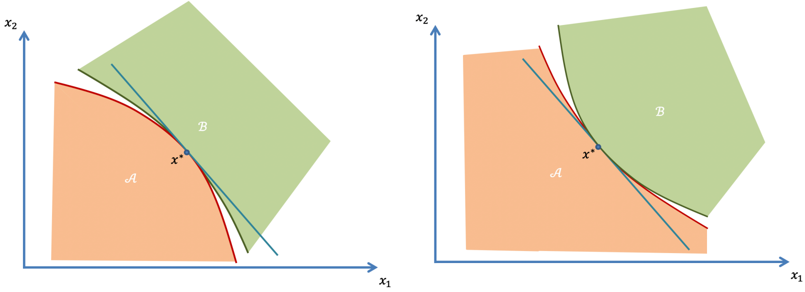
- Another remark is that only the relative prices matter for economic decisions.
- In our formulation here, nothing will change if we multiply vector p and related number b by same positive number.
- This result is consistent with our discussions in previous chapters.

Remark: Problems with Current Model

- Real life decentralization problem is more complicated.
- One problem is how correct price vector is found, since people may not have incentive to reveal their private information that is needed to calculate right prices.
- Besides, issues of externality and distribution arise when there are many producers and consumers.
- Interested students may refer to microeonomic textbooks.

Remark: Partial Failure of Decentralization

There exist cases where full decentralization is impossible.



(a) (b)

Remark: Partial Failure of Decentralization

i. In (a), $\mathcal{B}$ is not convex and x^* does not minimize the expenditure in the consumer problem. Here the consumer prefers extremes to a diversified bundle of goods.

ii. In (b), $\mathcal{A}$ is not convex and x^* does not maximize the producer's value of output. Here, the production technology has economies of scale or of specialization.

But in both cases, x^* maximizes $F(x)$ subject to $G(x) \leq c$.

Remark: Partial Failure of Decentralization

- For x^* to be a maximizer of the original problem, what really matters is relative curvature of F and G.

- We will discuss this idea and develop the conditions for maximization in Chapter 8.

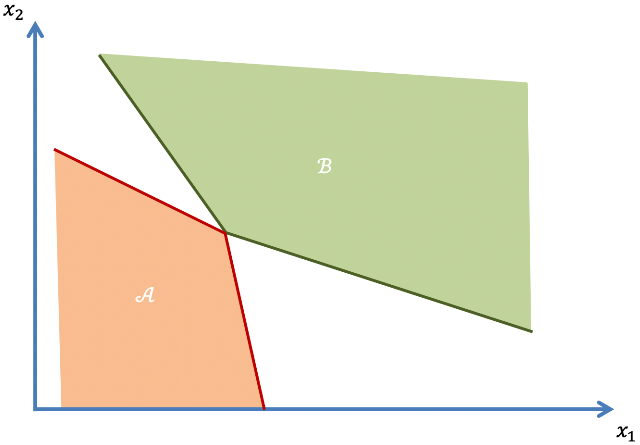
6.D. Uniqueness

- In Figure 6.1, the boundaries of the sets $\mathcal{B}$ and $\mathcal{A}$ are shown as smooth curves.

- But in general, a convex set can have straight-line segments along its boundary.

- Such possibilities have implications for separation and optimization.

Kinks

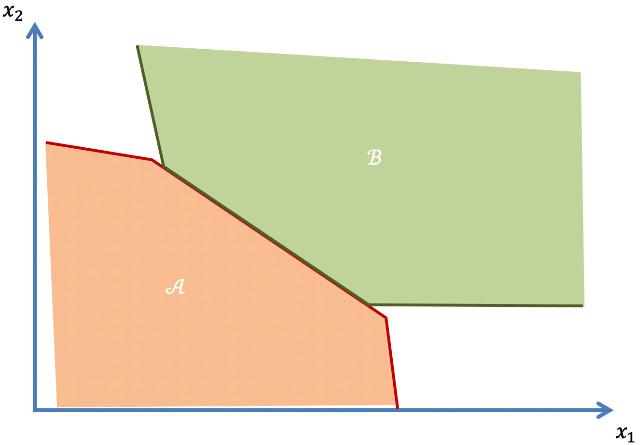


(a)

Kinks

- Two corners happen to meet at x^*.
- We can find many lines through x^* that separate the two sets: the decentralizing price vector p is not unique.
- It is not a serious problem for decentralization.
- In fact, separation is a more general notion than that of a common tangent.
- Decentralization depends on separation property.

Flat Portion



(b)

Flat Portion

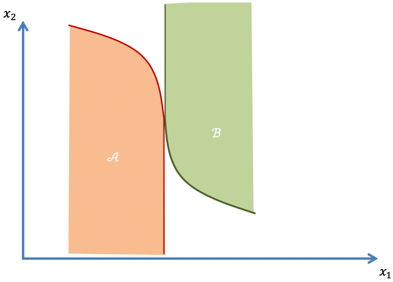
- Two sets have a flat portion in common.
- Any points along this region serves as the optimum x^*.
- It causes problems about decentralization.
- Given p, all points on the flat portion of $\mathcal{A}$ will yield same value of output to producer; and all those on the flat portion of $\mathcal{B}$ will yield same utility to consumer.

Flat Portion

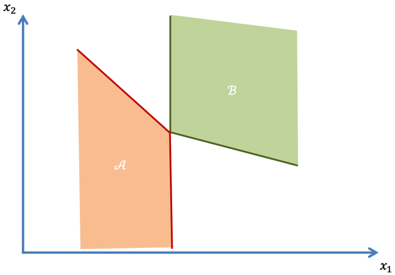
- Thus, there is no reason to believe that choices made separately by producer and consumer would coincide.

- In such a situation, we could only make a weaker claim: if producer and consumer happen to make coincident choices, neither will have strict incentive to depart from such choices.

Free Good



(c)



(d)

Free Good

- In (c), the two boundaries have vertical parts in common.
- We will have a vertical separating line, indicating $p_2 = 0$. In such cases, good 2 is a free good.
- Similarly, horizontal separating lines imply $p_1 = 0$, i.e., good 1 is a free good.
- In (d), there is a vertical separating line, whereas there are also non-vertical ones.
- Without stronger assumptions, it is not possible to guarantee strictly positive prices.

Negative Prices

- The case of a positive slope of the common tangent, or negative price of either good, is usually avoided by assuming
  - either that "free disposal" is possible so that the boundary of $\mathcal{A}$ cannot slope upward;
  - or that both goods are desirable so that the boundary of $\mathcal{B}$ cannot slope upward.
- In our figures, these assumptions are implicit.

Problems with Straight-line Segments

To summarize,

- The problems of kinks are not serious.
- In fact, such cases generalize the concept of tangency and preserve the decentralization property.
- Problems of flats are more serious because optimum choices can be non-unique and decentralization becomes problematic.

Solution to problems of flats

- We will then discuss what additional assumptions are needed to avoid this problem.

- In fact, a strengthening of the concepts of quasi-convaxity and quasi-convexity will suffice.

Strongly Convex Set

Definition 6.D.1 (Strongly Convex Set)**.** A set $\mathcal{S}$ of points in n-dimensional space is called strongly convex if, given any two points $x^a \in \mathcal{S}$ and $x^b \in \mathcal{S}$ and any real number $\alpha \in (0, 1)$, the point $\alpha x^a + (1 - \alpha)x^b$ is interior to $\mathcal{S}$.

Strictly Quasi-concave Function

Definition 6.D.2 (Strictly Quasi-concave Function)**.** A function $f : \mathcal{S} \to \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, is strictly quasi-concave if the set $\{x | f(x) \geq c\}$ is strongly convex for all $c \in \mathbb{R}$, or equivalently, if

$$f(\alpha x^a + (1-\alpha)x^b) > \min\{f(x^a), f(x^b)\},$$

for all x^a, x^b and for all $\alpha \in (0, 1)$.

Strictly Quasi-convex Function

Definition 6.D.3 (Strictly Quasi-convex Function)**.** A function $f : \mathcal{S} \to \mathbb{R}$, defined on a convex set $\mathcal{S} \subset \mathbb{R}^N$, is strictly quasi-convex if the set $\{x | f(x) \leq c\}$ is strongly convex for all $c \in \mathbb{R}$, or equivalently, if

$$f(\alpha x^a + (1-\alpha)x^b) < \max\{f(x^a), f(x^b)\},$$

for all x^a, x^b and for all $\alpha \in (0, 1)$.

Strictly Quasi-concave F

- Consider again the problem of maximizing $F(x)$ subject to $G(x) \leq c$, but now consider F being strictly quasi-concave, and G still being quasi-convex.

- Suppose x^* satisfies the conditions of Theorem 6.2.

- Then, x^* must be a unique solution.

Strictly Quasi-concave F

- To see this, we show by contradiction.
- Suppose that $\hat{x}$ is another solution.
- Then, x^* and $\hat{x}$ should be optimal for the consumer's problem.
- Thus, $px^* = p\hat{x} = b$ and $F(x^*) = F(\hat{x}) = v^*$.

Strictly Quasi-concave F

Consider the point $\tilde{x} = \alpha x^* + (1-\alpha)\hat{x}$, for some $\alpha \in (0,1)$.

(i) $p\tilde{x} = p(\alpha x^* + (1-\alpha)\hat{x}) = \alpha p x^* + (1-\alpha)p\hat{x}.$

$= \alpha b + (1 - \alpha b) = b$

(ii) Since F is strictly quasi-concave,

$$F(\tilde{x}) > \min\{F(x^*), F(\hat{x})\} = v^*.$$

Strictly Quasi-concave F

- By continuity of F, there exists $\beta < 1$, such that $F(\beta\tilde{x}) > v^*$ (i.e., $\beta\tilde{x}$ is interior to $\mathcal{B}$).
- Besides, $p(\beta\tilde{x}) < p\tilde{x} = b$.
- Thus, the bundle $\beta\tilde{x}$ is interior to $\mathcal{B}$ with $p(\beta\tilde{x}) < b$, contradicting with separation property.
- Therefore, initial supposition must be wrong.
- Strict quasi-concavity of F implies uniqueness of maximizer.

Strictly Quasi-convex G

Remark. Strict quasi-convexity of G together with quasi-concavity of F also imply the uniqueness of the maximizer. If there are more than one constraint, we require every component constraint function G^i to be strictly quasi-convex.

6.E. Examples

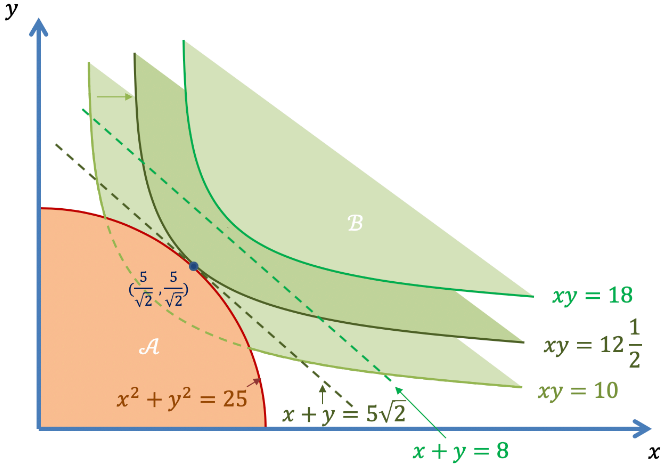
Example 6.1: Illustration of Separation

Consider the following problem:

$$\max_{x \geq 0,\, y \geq 0} F(x, y) = xy$$

$$\text{s.t. } G(x, y) = x^2 + y^2 \leq 25.$$

Example 6.1: Illustration of Separation



Example 6.2: Indirect Utility and Expenditure Functions

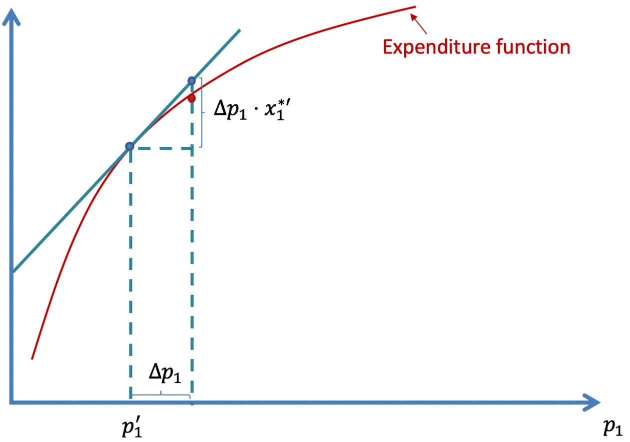
Part I: Expenditure Function

The expenditure function is

$$E(p,u) = \min_x \{px | U(x) \geq u\}.$$

Show that $E(p,u)$ is concave in p for each fixed u.

Example 6.2 (Part I): Intuition



Part II: Indirect Utility Function

The indirect utility function is

$$V(p, I) = \max_{x}\{U(x)|px \leq I\}.$$

Show that $V(p, I)$ is quasi-convex in (p, I).

Example 6.2 (Part II): Intuition