Chapter 4. Shadow Prices

4.A. Comparative Statics

The examination of a change in outcome in response to a change in underlying economic parameters is known as *comparative statics* analysis.

Take the consumer choice model as an example:

$$\max_{x \geq 0} U(x)$$
$$\text{s.t. } p \cdot x \leq I.$$

Here, the underlying economic parameters are the prices p and the income I. Suppose that the optimal choice is x^*. Then, we could perform the following *comparative static* analysis.

Income effect:

- Good l is *normal* if x_l^* is increasing in I;
- Good l is *inferior* if x_l^* is decreasing in I.

Price effect:

- Good l is a *regular good* if x_l^* is decreasing in p_l.
- Good l is a *Giffen good* if x_l^* is increasing in p_l. (Example: potatoes at low income level)
- Good l is a *gross substitute* for Good k if x_l^* is increasing in p_k.
- Good l is a *gross compelement* for Good k if x_l^* is decreasing in p_k.

In Chapter 1, we have learned the concept of *Marginal Utility of Income*, namely, the marginal increase of utility induced by a marginal change of income. This is also a *comparative static* result. We have also learned that the value of *Marginal Utility of Income* is the Lagrange multiplier λ. Therefore, it seems that the Lagrange multiplier λ has an important economic meaning, and provides the answer to a particular type of comparative static questions. In this chapter, we will focus on λ in general settings.

4.B. Equality Constraints

In this section, we will discuss the meaning of Lagrange multipliers for the equality cosntraints. We will first discuss the special case of two-good consumer choice model, and then move on to the general case with two variables and one constriant. At last, we will consider more variables and more constraints.

Marginal Utility of Income. We start with a simple two-good consumer choice model. Recall Example 2.1:

> Consider a consumer choosing between two goods x and y, with prices p and q respectively. His income is I, so the budget constraint is $px + qy = I$.
> The utility function is $U(x, y) = \alpha \ln(x) + \beta \ln(y)$.

We have solved the problem in Chapter 2 and the solution is

$$x^* = \frac{\alpha I}{(\alpha+\beta)p},\ y^* = \frac{\beta I}{(\alpha+\beta)q},\ \lambda = \frac{(\alpha+\beta)}{I}.$$

Now we are interested to know the effect of the extra amount $\mathrm{d}I$ of income on the maximum utility $U(x^*, y^*)$.

One way to solve this problem is to write the maximum utility as a function of I and differentiate it with respect to I directly. To calculate the maximum utility, we plug the resulting optimal consumption bundle into the objective function:

$$V(p,q,I) = U(x^*, y^*) = \alpha \ln(x^*) + \beta \ln(y^*) = \alpha \ln\left(\frac{\alpha I}{(\alpha+\beta)p}\right) + \beta \ln\left(\frac{\beta I}{(\alpha+\beta)q}\right).$$

In microeconomic theory, the maximum utility function $V(p, q, I)$ is called the *indirect utility function*, to distinguish it from the *direct utility function* $U(x, y)$ which is defined directly over the consumption bundle.

With the explicit representation of $V(p, q, I)$, we could calculate the marginal change of maximum utility $V(p, q, I)$ with respect to the marginal change of I directly, as follows:

$$\frac{\partial V(p,q,I)}{\partial I} = \alpha \frac{(\alpha+\beta)p}{\alpha I}\frac{\alpha}{(\alpha+\beta)p} + \beta \frac{(\alpha+\beta)q}{\beta I}\frac{\beta}{(\alpha+\beta)q} = \frac{(\alpha+\beta)}{I}.$$

Comparing the value of $\frac{\partial V(p,q,I)}{\partial I}$ with the value of λ, it is not hard to see that they are the same. Therefore, we could have known the utility increment per unit of marginal addition to income, or *Marginal Utility of Income*, without calculating $\frac{\partial V(p,q,I)}{\partial I}$ directly.

The result should not be too surprising. In Chapter 1, we have already mentioned that the economic meaning of the Lagrange multiplier λ in the consumer choice model is the *Marginal Utility of Income*.

Below, we reiterate the argument in Chapter 1. First, we write out the problem properly as follows:

$$V(p_1, p_2, I) = \max_{x_1, x_2 \geq 0} U(x_1, x_2)$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 = I.$$

Then, we want to argue that λ represents the change of (maximum) utility with respect to a marginal change of income.

The argument proceeds as follows:

(i) Suppose that we have an interior solution, then the consumer would be indifferent between spending the extra amount $\mathrm{d}I$ of income on good 1 or good 2.

To see this, spending the additional income on good 1 gives additional $MU_1 \mathrm{d}I/p_1$ units of utility and spending on good 2 gives additional $MU_2 \mathrm{d}I/p_2$ units of utility. We could show the equivalence of the two utility increments, or MU_1/p_1 and MU_2/p_2, by the first-order necessary conditions. The Lagrangian of the problem is

$$\mathcal{L}(x, \lambda) = U(x_1, x_2) + \lambda(I - p_1 x_1 - p_2 x_2.)$$

The first-order necessary conditions on x_1 and x_2 suggest $\lambda = MU_1/p_1 = MU_2/p_2$. We have established that spending the additional income on good 1 and on good 2 have the same effect, and we further know that the effect could also be represented by λ. That is, $\mathrm{d}I$ will increase the utility by $MU_1 \mathrm{d}I/p_1 = MU_2 \mathrm{d}I/p_2 = \lambda \, \mathrm{d}I$.

(ii) Suppose otherwise, that one of the goods attains a corner solution, say $x_2^* = 0$. Then, by the first-order necessary conditions, we know $\lambda = MU_1/p_1 \geq MU_2/p_2$.

Therefore, spending dI on good 1 gives weakly more utility increment, that is, $MU_1 \mathrm{d}I/p_1 \geq MU_2 \mathrm{d}I/p_2$, and the utility increment is again equal to $\lambda\,\mathrm{d}I$. In the end, the consumer would spend the additional income solely on good 1 if the inequality holds strict; and she would be indifferent between spending on either good when the inequality holds with equality. In any case, the utility increment could be represented by $\lambda\,\mathrm{d}I$.

Hopefully, you are now convinced that λ represents the *martinal utility of income* $\mathrm{d}v/\mathrm{d}I$. We will now move to the general case with two variables and one constraint.

Two variables, one constraint. In the following discussions, we assume that the choice variables attain **interior solutions**, or that we do not impose any non-negativity constraints. However, you should keep in mind that the result extends to the situations where the choice variables attain corner solutions, just as our previous argument for *marginal utility of income* (argument (ii)) shows. Similar arguments should go through when you do obtain a corner solution.

The maximization problem is

$$v = \max_{x_1, x_2} F(x_1, x_2) \qquad \text{(MP1)}$$
$$\text{s.t. } G(x) = c.$$

We are interested to know how much the highest attainable value v would increase due to a marginal addition to c. And we claim that the Lagrange multiplier λ presents this value. We already kwow from our previous argument on the *marginal utility of income* that the claim is right when $G(x)$ is linear. Below, we will show that the claim is in general correct.

First, we need to introduce as few more notations. Suppose c increases by an infinitesimal amount dc. So, the maximization problem becomes

$$v + \mathrm{d}v = \max_{x_1, x_2} F(x_1, x_2) \qquad \text{(MP2)}$$
$$\text{s.t. } G(x) = c + \mathrm{d}c.$$

$v + \mathrm{d}v$ represents the new optimum value. We follow notations in the previous chapters and define the solution to (MP1) $x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$. We further define $x^* + \mathrm{d}x^* = \begin{pmatrix} x_1^* + \mathrm{d}x_1^* \\ x_2^* + \mathrm{d}x_2^* \end{pmatrix}$ to be the solution to our new maximization problem (MP2). Note that $\mathrm{d}x^*$ is not arbitrary;[1] it is the *optimum* small change in the choice, arising in response to a small change in c. Next, we will be able to derive the result.

$$
\begin{aligned}
\mathrm{d}v \underbrace{=}_{\text{by definition}} (v + \mathrm{d}v) - v \underbrace{=}_{\text{by definition}} F(x^* + \mathrm{d}x^*) - F(x^*) \underbrace{=}_{\text{Taylor approximation}} F_1(x^*)\mathrm{d}x_1^* + F_2(x^*)\mathrm{d}x_2^* \\
\underbrace{=}_{\text{First-order conditions}} \lambda G_1(x^*)\mathrm{d}x_1^* + \lambda G_2(x^*)\mathrm{d}x_2^* = \lambda \left[G_1(x^*)\mathrm{d}x_1^* + G_2(x^*)\mathrm{d}x_2^*\right] \\
\underbrace{=}_{\text{Taylor approximation}} \lambda \left[G(x^* + \mathrm{d}x^*) - G(x^*))\right] \underbrace{=}_{\text{constraints}} \lambda \left[(c + \mathrm{d}c) - c\right] = \lambda \, \mathrm{d}cs
\end{aligned}
$$

The result could be written as follows:

$$
\mathrm{d}v/\mathrm{d}c = \lambda. \tag{4.1}
$$

Thus, *the Lagrange multiplier is the rate of change of the maximum attainable value of the objective function with respect to a change in the parameter on the right-hand side of the constraint.*

More variables and more constraints. Now, we consider the case with more variables and more constraints. The maximization problem is

$$
\begin{aligned}
v = \max_{x_1, x_2, \ldots, x_n} F(x_1, x_2, \ldots, x_n) \\
\text{s.t. } G^1(x) = c_1, G^2(x) = c_2, \ldots, G^m(x) = c_m.
\end{aligned} \tag{MP3}
$$

In matrix notation, it is

$$
\begin{aligned}
v = \max_x F(x) \\
\text{s.t. } G(x) = c.
\end{aligned} \tag{MP3'}
$$

We first consider a change of only one constraint. Suppose, say, c_1 increases by an infinitesimal amount $\mathrm{d}c_1$.[2]

[1] Previously, we used $\mathrm{d}x$ to denote arbitrary deviations when deriving *first-order necessary conditions*.

[2] You will see that the calculation for a change of only one constraint is no simpler than the calculation

The maximization problem becomes

$$v + \mathrm{d}v = \max_{x_1, x_2, ..., x_n} F(x_1, x_2, ..., x_n) \tag{MP4}$$
$$\text{s.t. } G^1(x) = c_1 + \mathrm{d}c_1, G^2(x) = c_2, ..., G^m(x) = c_m.$$

Again, $v + \mathrm{d}v$ represents the new optimum value. And we denote the solution to (MP3) as $x^* = \begin{pmatrix} x_1^* \\ \vdots \\ x_n^* \end{pmatrix}$. We further define $x^* + \mathrm{d}x^* = \begin{pmatrix} x_1^* + \mathrm{d}x_1^* \\ \vdots \\ x_n^* + \mathrm{d}x_n^* \end{pmatrix}$ to be the solution to our new maximization problem (MP4). Note that even though only one constraint changes, we need to reoptimize and all x_j^* might change.

Next, we derive the result.

$$\begin{aligned}
\mathrm{d}v &\underbrace{=}_{\text{by definition}} (v + \mathrm{d}v) - v \underbrace{=}_{\text{by definition}} F(x^* + \mathrm{d}x^*) - F(x^*) \underbrace{=}_{\text{Taylor approximation}} F_1(x^*)\mathrm{d}x_1^* + ... + F_n(x^*)\mathrm{d}x_n^* \\
&\underbrace{=}_{\substack{\uparrow \\ \text{first-order conditions}}} \sum_{i=1}^{m} \left[\lambda_i G_1^i(x^*)\right] \mathrm{d}x_1^* + ... + \sum_{i=1}^{m} \left[\lambda_i G_n^i(x^*)\right] \mathrm{d}x_n^* \\
&= \sum_{i=1}^{m} \left[\lambda_i G_1^i(x^*)\mathrm{d}x_1^*\right] + ... + \sum_{i=1}^{m} \left[\lambda_i G_n^i(x^*)\mathrm{d}x_n^*\right] \\
&= \sum_{j=1}^{n} \sum_{i=1}^{m} \left[\lambda_i G_j^i(x^*)\mathrm{d}x_j^*\right] = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\lambda_i G_j^i(x^*)\mathrm{d}x_j^*\right] \\
&= \sum_{i=1}^{m} \left\{\lambda_i \sum_{j=1}^{n} \left[G_j^i(x^*)\mathrm{d}x_j^*\right]\right\} \underbrace{=}_{\substack{\uparrow \\ \text{Taylor approximation}}} \sum_{i=1}^{m} \left\{\lambda_i \left[G^i(x^* + \mathrm{d}x^*) - G^i(x^*)\right]\right\} \\
&\underbrace{=}_{\substack{\uparrow \\ \text{constraints}}} \sum_{i=1}^{m} \{\lambda_i [c_i + \mathrm{d}c_i - c_i]\} = \sum_{i=1}^{m} \lambda_i \mathrm{d}c_i \qquad (4.2) \\
&\underbrace{=}_{\text{change in } c_1 \text{ only}} \lambda_1 \,\mathrm{d}c_1.
\end{aligned}$$

Therefore, $\mathrm{d}v = \lambda_1 \mathrm{d}c_1$ if we only consider a marginal change in c_1 and remain unchanged all the other constraints.

Observe Equation (4.2). It shows that we already obtained the result for simultaneous changes of multiple constraints: $\mathrm{d}v = \sum_{i=1}^{m} \lambda_i \,\mathrm{d}c_i$.

for changes in many constraints.

If you are familiar with the vector-matrix notation, the calculation is much simpler:

$$\mathrm{d}v \underbrace{=}_{\text{by definition}} (v + \mathrm{d}v) - v \underbrace{=}_{\text{by definition}} F(x^* + \mathrm{d}x^*) - F(x^*) \underbrace{=}_{\text{Taylor approximation}} F_x(x^*)\mathrm{d}x^*$$

$$\underbrace{=}_{\text{First-order conditions}} \lambda G_x(x^*)\mathrm{d}x^* \underbrace{=}_{\substack{\text{Taylor}\\\text{approximation}}} \lambda \left[G(x^* + \mathrm{d}x^*) - G(x^*)\right] \underbrace{=}_{\text{constraints}} \lambda \left[(c + \mathrm{d}c) - c\right] = \lambda\, \mathrm{d}c$$

The final result is $\mathrm{d}v = \lambda\, \mathrm{d}c$, where λ is a m-dimansional row vector and $\mathrm{d}c$ is a m-dimensional column vector. This result coincide with Equation (4.2).

This result is important and thus summarized below:

Result (Interpretation of Lagrange Multipliers)**.**

If

$$v = \max_x F(x)$$
$$s.t.\ \ G(x) = c.$$

and λ *is the row vector of multipliers for the constraints, then change* $\mathrm{d}v$ *that results from an infinitesimal change* $\mathrm{d}c$ *is given by*

$$\mathrm{d}v = \lambda\, \mathrm{d}c. \tag{4.3}$$

4.C. Shadow Prices

Marginal Product of Labor. To illustrate and explain (4.3), consider a planned economy for which a production plan x^* is to be chosen to maximize a social welfare function $F(x)$. The vector of the plan's resources requirement is $G(x)$, and the vector of the available amounts of these resources is c. That is,

$$v = \max_x \underbrace{F(x)}_{\text{social welfare function}} \tag{MP5}$$

$$\text{s.t. } \underbrace{G(x) = c}_{\text{resource constraints}}.$$

Assume that the first constraint $G^1(x) = c_1$ is labor constraint. Suppose the problem has been solved and the vector of Lagrange multipliers λ is known.

Now, suppose some power outside the economy puts a small additional amount $\mathrm{d}c_1$ of labor into the economy. We know from the previous analysis that without further calculation, we already know the resultant increase in social welfare, which is simply $\lambda_1 \, \mathrm{d}c_1$. We can then say that the Lagrange multiplier λ_1 is the *marginal product of labor* in this economy, measured in units of its social welfare.

Demand price. Now suppose that the additional labor can only be used at some cost. The maximum the economy is willing to pay in terms of its social welfare units is λ_1 per marginal unit of c_1. In this natural sense, the Lagrange multiplier λ_1 is the *demand price* the planner places on labor services.

You may find a price expressed in units of social welfare strange. The critic makes sense, however, the more important indicator is the relative demand prices of different resources, rather than the absolute demand prices of single resources, since the relative demand prices govern the economy's willingness to exchange one resource for another. To see this, assume that the second constraint $G^2(x) = c_2$ is land constraint. Now, we are interested to know how much land the economy is willing to give up for an additional $\mathrm{d}c_1$ of labor. Assume the amount of land to give up is $\mathrm{d}c_2$. Then the net gain in social welfare from this transaction is $\lambda_1 \, \mathrm{d}c_1 - \lambda_2 \, \mathrm{d}c_2$. Therefore, the most land the economy is willing to give up is $\lambda_1/\lambda_2 \, \mathrm{d}c_1$.[3]

The relative demand prices is very relevant to the theory of international trade. We will not go deep into this topic. But the simple intuition is that if a neighboring economy has a different trade-off between two resources, then there is a possibility of mutually advantageous trade.[4]

"Invisible Hand". Now, we will discuss the link between market prices and Lagrange multipliers. Consider an economy that allocates resources using market. In equilibrium, the prices are determined by supplies and demands. And suppose that an economist works out a planner's problem (MP5) and gets a vector of Lagrange multipliers for the

[3] The ratio λ_1/λ_2, or the relative demand price of labor and land, is the demand price of a unit of labor measured in units of land.

[4] The trade could be directly on the factors, or indirectly through goods made of these factors.

resource constraints. The social welfare function could be viewed as the criterion to evaluate the performance of the economy.

The question is whether the market economy could replicate the planned allocation, which is the best allocation for a given criterion. There are in fact important cases where the optimum can be replicated in the market, and the Lagrange multipliers are proportional to the market prices of the resources: the relative prices equal the corresponding ratios of multipliers. In such cases, the economist would say that the economy is guided by an "*invisible hand*" to his planned optimum. Example 4.1 is such a case. The concepts become clearer when we get there.

To evoke the connection with prices, and yet maintain a conceptual distinction from market prices, Lagrange multipliers are often called *shadow prices*.

4.D. Inequality Constriants

So far, in our analysis, we assumed that the constraints are equality constraints. In economic applications, it is reasonable to consider inequality constraints. As we discussed in Chapter 3, full employment of resources may not be optimal. We have seen a case of *Technological Unemployment* in Example 3.2. In fact, the study of inequality constraints also turns out to be important in understanding the meaning of Lagrange multipliers λ. The main problem with equality constraints is that because of the connection between prices and *shadow prices* (the Lagrange multipliers), we do expect the Lagrange multipliers to be non-negative, however, the maximization problems with equality constraints do not impose any restrictions on the sign of λ. The reason is that for equality constraints, an increase in the right-hand side of a constraint equation does not necessarily mean a relaxation of the constraint. More specifically, the equality constraint $G^i(x) = c_i$ could be written as $-G^i(x) = -c_i$. An increase in $-c_i$ would be a decrease in the quantity c_i of resource i. Such problems could be avoided if we write the constraints as inequality constraints.

We will now study the inequality constraints. The problem under concern is:

$$v = \max_x \underbrace{F(x)}_{\text{social welfare function}} \tag{MP6}$$
$$\text{s.t.} \quad \underbrace{G(x) \le c}_{\text{resource constraints}}.$$

For inequality constraints, we invoke Kuhn-Tucker Theorem. The first-order necessary conditions on x_j's are still valid. Therefore, we could repeat our analysis for equality constraints, up until the point where constraints come into play:

$$\begin{aligned} \mathrm{d}v &\underbrace{=}_{\text{by definition}} (v + \mathrm{d}v) - v \underbrace{=}_{\text{by definition}} F(x^* + \mathrm{d}x^*) - F(x^*) \underbrace{=}_{\text{Taylor approximation}} F_x(x^*)\mathrm{d}x^* \\ &\underbrace{=}_{\text{First-order conditions}} \lambda G_x(x^*)\mathrm{d}x^* \underbrace{=}_{\text{Taylor approximation}} \lambda \left[G(x^* + \mathrm{d}x^*) - G(x^*)\right]. \end{aligned} \tag{4.4}$$

If the constraints are binding for x^* and continue to be binding for $x^* + \mathrm{d}x^*$, we could complete the analysis as we did for the equality constraints. That is, following (4.4),

$$\mathrm{d}v = ... = \lambda \left[G(x^* + \mathrm{d}x^*) - G(x^*)\right] = \lambda \left[(c + \mathrm{d}c) - c\right] = \lambda \,\mathrm{d}c.$$

Whether the constraints are binding is related to the first-order neccessary conditions for λ. We now check the conditions.

The first-order necessary conditions for λ give:

$$\mathcal{L}_\lambda(x^*, \lambda) = c - G(x^*) \ge 0, \;\; \lambda \ge 0, \;\text{ with complementary slackness.}$$

The above conditions ensure non-negative Lagrange multipliers λ. This is the desirable property that we expect: *shadow prices* λ are non-negative.

Next, we will investigate into the first-order conditions on λ more carefully. Complementary slackness means that, for every i, at least one of the pair

$$G^i(x^*) \le c_i \text{ and } \lambda_i \ge 0$$

holds with equality.

That is,

(i) If resource i is not fully employed ($G^i(x^*) < c_i$), then its shadow price is zero ($\lambda_i = 0$).

(ii) If a resource is with a positive shadow price $\lambda_i > 0$, then it must be fully employed ($G^i(x^*) = c_i$).

This supports and completes the interpretation of shadow prices as the marginal value products of the resources. If part of some resource is already idle, then any increment in it will also be left idle. The maximum value of the objective function will not change, and the shadow price will be zero. On the other hand, a positive shadow price means that a marginal increment in resource availability can be put to good use. Then none of the amount originally available can have been left idle in the original plan.

There is one tricky point. Suppose that c_i is such that resource i is fully used ($G^i(x^*) = c_i$), but any increment will be left unused ($G^i(x^{**}) = c_i < c_i + \mathrm{d}c_i$), where x^{**} denotes the optimal outputs after the increment of $\mathrm{d}c_i$. Complementary slackness does not tell us whether the multiplier will be positive or zero at this point. In fact, both cases could happen. Which case would happen depends on whether the slope of the maximum value v as a function of c_i drops smoothly or suddenly to 0. Figure 4.1 and 4.2 below show both possibilities.

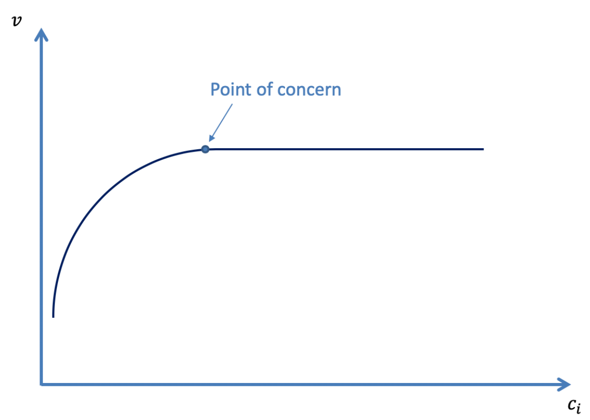

Figure 4.1: $\lambda_i = 0$

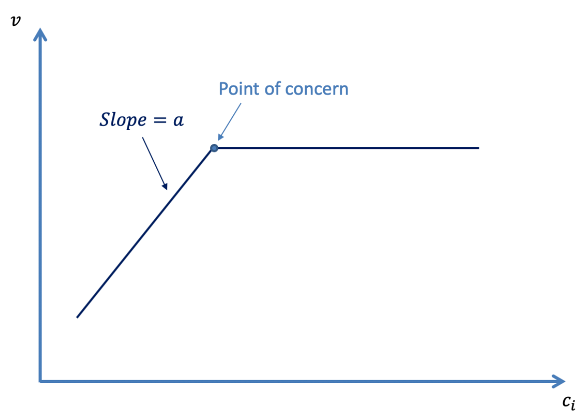

Figure 4.2: $\lambda_i \in [0, a]$

4.E. Examples

Example 4.1: The Invisible Hand - Distribution. Consider the stage of planning where the production of the various goods is already known, and the only remaining question is that of distributing them among the consumers. There are C consumers, labeled $c = 1, 2, ..., C$, and G goods, labeled $g = 1, 2, ..., G$. Let X_g be the fixed total amount of good g, and x_{cg} the amoung allocated to consumer c. Each consumer's utility is a function only of his own allocation:

$$u_c = U^c(x_{c1}, x_{c2}, ..., x_{cG}). \tag{4.5}$$

Social welfare is a function of these utility levels:

$$w = W(u_1, u_2, ..., u_C).$$

Assume that the utilities and social welfare function are increasing functions in their respective arguments. Assume also that at the social optimum $x^*_{cg} > 0$ for all c and g. The constriants are

$$x_{1g} + x_{2g} + ... + x_{Cg} \leq X_g, \text{ for } g = 1, 2, ..., G. \tag{4.6}$$

Question 1: Write down the first-order conditions for the socially optimal allocation.

Question 2: Now suppose the Lagrange multipliers, or *shadow prices*, are made the prices of the goods. Every consumer c is given a money income I_c and allowed to choose his consumption bundle to maximize his utility (4.5) subject to his budget cosntraint. Show that by adjusting money income I_c, the social optimum is attainable. This is when the distribution of income is such that at the margin the social value of every consumer's income is the same. The attainment of the social optimum in the decentralized problem is the "invisible hand" result for the distribution problem.

Solution.

Question 1: First, since the utilities and social welfare function are increasing functions in their respective arguments, no goods are going to be wasted, so we can express the constriants as equations. We formally state the problem as follows:

$$\max_{\substack{x_{11},x_{12},...,x_{1G}\\x_{21},x_{22},...,x_{2G}\\...\\x_{C1},x_{C2},...,x_{CG}}} W\left(U^1(x_{11},...,x_{1G}),U^2(x_{21},...,x_{2G}),...,U^C(x_{C1},...,x_{CG})\right)$$
$$\text{s.t. } x_{1g}+x_{2g}+...+x_{Cg}=X_g, \text{ for } g=1,2,...,G.$$

This is a maximization problem with equality constraints. To solve this problem, we invoke Lagrange's Theorem.

i Form Lagrangian:

$$\mathcal{L}(x,\pi)=W\left(U^1(x_{11},...,x_{1G}),U^2(x_{21},...,x_{2G}),...,U^C(x_{C1},...,x_{CG})\right)+\sum_{g=1}^{G}\pi_g\left[X_g-\sum_{c=1}^{C}x_{cg}\right].$$

ii First-order conditions:

$$\partial\mathcal{L}/\partial x_{cg}=(\partial W/\partial u_c)\left(\partial U^c/\partial x_{cg}\right)-\pi_g=0, \tag{4.7}$$
$$\text{for all } c=1,...,C \text{ and } g=1,...,G;$$
$$\partial\mathcal{L}/\partial\pi_g=X_g-\sum_{c=1}^{C}x_{cg}=0 \text{ for all } g=1,...,G.$$

All the partial derivatives are to be evaluated at the optimum.

Question 2: Now, π_g are made prices of the goods. For any consumer c, the budget constraint is

$$\pi_1x_{c1}+\pi_2x_{c2}+...+\pi_Gx_{cG}\le I_c.$$

Again, because the utilities are increasing functions, the budget constraints should hold with equality:

$$\pi_1x_{c1}+\pi_2x_{c2}+...+\pi_Gx_{cG}=I_c.$$

We formally state the consumer's problem:

$$\max_{x_{c1},x_{c2},...,x_{cG}} U^c(x_{c1},...,x_{cG})$$
$$\text{s.t. } \pi_1 x_{c1} + \pi_2 x_{c2} + ... + \pi_G x_{cG} = I_c.$$

This is a maximization problem with equality constraints. To solve this problem, we invoke Lagrange's Theorem.

i Form Lagrangian:

$$\mathcal{L}(x, \lambda_c) = U^c(x_{c1},...,x_{cG}) + \lambda_c \left[I_c - \pi_1 x_{c1} - \pi_2 x_{c2} - ... - \pi_G x_{cG} \right].$$

ii First-order conditions:

$$\partial\mathcal{L}/\partial x_{cg} = (\partial U^c/\partial x_{cg}) - \lambda_c \pi_g = 0, \text{ for all } g = 1,...,G; \tag{4.8}$$
$$\partial\mathcal{L}/\partial\lambda_c = I_c - \pi_1 x_{c1} - \pi_2 x_{c2} - ... - \pi_G x_{cG} = 0.$$

The above conditions hold for every consumer c.

Our objective is to achieve the social optimum, that is, we want the conditions (4.7) and (4.8) to coincide. This happens when

$$\partial W/\partial u_c = 1/\lambda_c, \text{ or } (\partial W/\partial u_c)\, \lambda_c = 1, \text{ for all } c. \tag{4.9}$$

This can be done by adjusting the income I_c. The left-hand side of the second equation in (4.9) is simply the marginal effect on social welfare of giving a unit of income to consumer c; it is the marginal effect on c's own utility times the effect of a unit of his utility on social welfare. The right-hand side is a constant. Therefore, the social optimum could be attained when the distribution of income is arranged so that at the margin the social value of every consumer's income is the same.

Remark 1. The argument comparing first-order conditions is not fully rigorous.

Remark 2. The crucial assumption leading to the result is the independence of every consumer's utility on any other consumer's consumption.

Example 4.2: Duty-Free Purchases. Imagine the consumption decision of a jet-setter. He can buy various brands of liquor at his home-town store, or at the duty-free stores of the various airports he travels through. The duty-free stores have cheaper prices, but the total quantity he can buy there is restricted by his home country's customs regulations. There are n brands. Let p be the row vector of home-town prices and q that of duty-free prices. The duty-free prices are uniformly lower: $q \ll p$. Let x be the column vector of his home-town purchases and y that of the duty-free. Assume that the quantities as continuous variables. Suppose during the year, only total dollar amount M of duty-free liquor is allowed, that is,

$$q_1 y_1 + q_2 y_2 + ... + q_n y_n \leq K.$$

The jet-setter's total consumption is $c = x + y$, and he derives utility $U(c)$ from liquor consumption.

Also assume that the income allocated to liquor consumption is fixed at I. Thus, the budget constraint is

$$px + qy \leq I.$$

How much liquor should be jet-setter buy, and from which source?

Solution. We formally state the problem as follows:

$$\begin{aligned} &\max_{x,y} U(x+y) \\ \text{s.t. } & y_1 + y_2 + ... + y_n \leq K \\ & px + qy \leq I \\ & x \geq 0, y \geq 0. \end{aligned}$$

This is a maximization problem with inequality constraints. To solve this problem, we invoke Kuhn-Tucker Theorem.

i Form Lagrangian:

$$\mathcal{L}(x, y, \lambda, \mu) = U(x+y) + \lambda \left[I - px - qy\right] + \mu \left[K - y_1 - y_2 - ... - y_n\right].$$

ii First-order conditions:

$$\partial\mathcal{L}/\partial x_j = (\partial U/\partial c_j) - \lambda p_j \leq 0, x_j \geq 0, \text{ with complementary slackness} \quad (4.10)$$

for all $j = 1, ..., n$;

$$\partial\mathcal{L}/\partial y_j = (\partial U/\partial c_j) - \lambda q_j - \mu \leq 0, y_j \geq 0, \text{ with complementary slackness} \quad (4.11)$$

for all $j = 1, ..., n$;

$$\partial\mathcal{L}/\partial\lambda = I - px - qy \geq 0, \lambda \geq 0, \text{ with complementary slackness;}$$

$$\partial\mathcal{L}/\partial\mu = K - y_1 - y_2 - ... - y_n \geq 0, \mu \geq 0, \text{ with complementary slackness.}$$

The inequality pairs permit 2^{2n+2} patterns of equations, and sorting them out systematically is hopeless. But a search assisted by economic intuition quickly reveals the solution. First, the budget constraint should hold with equality: any income left could have been spent to increase the utility. And if the quota constraint is slack, then the jet-setter could satisfy all his liquor needs from the cheaper duty-free stores and the problem becomes trivial. Suppose, more interestingly, we are in a situation in which the jet-setter is not satiated within his duty-free allowance, that is, the quota constraint is binding. We are still left with the 2^{2n} complementary slackness pairs for x_j's and y_j's.

We will further apply our economic intuition. We ask the question: can some brand j be bought in a positive amounts at both kinds of stores? If so, since $x_j^* > 0$ and $y_j^* > 0$, from (4.10) and (4.11),

$$(\partial U/\partial c_j) - \lambda p_j = 0 = (\partial U/\partial c_j) - \lambda q_j - \mu,$$

$$\text{or } \lambda p_j = \partial U/\partial c_j = \lambda q_j + \mu. \quad (4.12)$$

(4.12) is most likely to hold for at most one j. To see this, suppose otherwise, (4.12) holds for both $j = 1$ and 2,

$$\lambda p_1 = \partial U/\partial c_1 = \lambda q_1 + \mu \text{ and } \lambda p_2 = \partial U/\partial c_2 = \lambda q_2 + \mu$$

$$\Longrightarrow \lambda(p_1 - q_1) = \mu = \lambda(p_2 - q_2).$$

Since the consumer is not satiated, that is, a relaxation of the budget constraint would translate into an increase in utility, λ is positive.

Then,

$$p_1 - q_1 = p_2 - q_2.$$

With given prices, this can occur only by chance.

Now suppose brand j is bought only in the duty-free store. Then, since $x_j^* = 0$ and $y_j^* > 0$, from (4.10) and (4.11), we have

$$(\partial U/\partial c_j) \leq \lambda p_j \tag{4.13}$$

$$\text{and} \;\; (\partial U/\partial c_j) = \lambda q_j + \mu. \tag{4.14}$$

The left-hand side of both (4.13) and (4.14) are the marginal utility of brand j. The right-hand side of (4.13) is the marginal opportunity cost of buying it at the home-town store: to do so takes p_j of income which can not then be used for other purchases, and the utility value of this amount of income is λp_j. The brand is not bought at the home-town store since the marginal opportunity cost of purchasing at the home-town store exceeds the marginal utility of brand j. The right-hand side of (4.14) is the marginal opportunity cost of buying it at the duty-free store: this requires q_j of income having utility value λq_j and further it uses up a unit of the duty-free allowance, which has the shadow price μ. Note that we could deduct from (4.13) and (4.14) that the duty-free store has a lower opportunity cost: $\lambda q_j + \mu \leq \lambda p_j$. From the previous analysis, the inequality holds with equality for at most one j. The reverse is also true: if j is bought only in the home-town store, then we could deduct from the first-order necessary conditions that the home-town store has a lower opportunity cost.

Now the principle is clear: buy each brand at the outlet with the lower opportunity cost. Since $\lambda q_i + \mu < \lambda p_i$ if and only if $p_i - q_i > \mu/\lambda$, the jet-setter should rank the brands by their absolute price differences in the two kinds of stores. The brands with the largest price differences are bought at the duty-free stores, and those with the smallest price differences are bought at the home-town store. The meeting point of the two is chosen so as to use up the duty-free allowance. There may be at most one brand that is bought at both kinds of stores.