Chapter 4. Shadow Prices

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4.A. Comparative Statics

The examination of a change in outcome in response to a change in underlying economic parameters is known as *comparative statics* analysis.

Comparative Statics

Take the consumer choice model as an example:

$$\max_{x \geq 0} U(x)$$

$$\text{s.t. } p \cdot x \leq I.$$

Here, the underlying economic parameters are the prices p and the income I.

Comparative Statics: x^*

Income effect:

- Good l is *normal* if x_l^* is increasing in I;
- Good l is *inferior* if x_l^* is decreasing in I.

Comparative Statics: x^*

Price effect:

- Good l is a *regular good* if x_l^* is decreasing in p_l.
- Good l is a *Giffen good* if x_l^* is increasing in p_l.

  (Example: potatoes at low income level)
- Good l is a *gross substitute* for Good k if x_l^* is increasing in p_k.
- Good l is a *gross compelement* for Good k if x_l^* is decreasing in p_k.

Comparative Statics: $U(x^*)$

- In Chapter 1, we have learned the concept of *Marginal Utility of Income*, namely, the marginal increase of utility induced by a marginal change of income.

- We have also learned that the value of *Marginal Utility of Income* is the Lagrange multiplier λ.

- In this chapter, we will focus on λ in general settings.

4.B. Equality Constraints

- In this section, we will discuss the meaning of Lagrange multipliers for the equality cosntraints.

- We will first discuss the special case of two-good consumer choice model, and then move on to the general case with two variables and one constriant.

- At last, we will consider more variables and more constraints.

Marginal Utility of Income

We start with a simple two-good consumer choice model.

Recall Example 2.1:

> Consider a consumer choosing between two goods x and y, with prices p and q respectively. His income is I, so the budget constraint is $px + qy = I$.
>
> The utility function is $U(x, y) = \alpha \ln(x) + \beta \ln(y)$.

Marginal Utility of Income

We have solved the problem in Chapter 2:

$$x^* = \frac{\alpha I}{(\alpha+\beta)p},\ y^* = \frac{\beta I}{(\alpha+\beta)q},\ \lambda = \frac{(\alpha+\beta)}{I}.$$

Question: what is the effect of the extra amount $\mathrm{d}I$ of income on the maximum utility $U(x^*, y^*)$?

Marginal Utility of Income

One way to solve this problem is

(i) Write the maximum utility as a function of I:

$$\begin{aligned} V(p,q,I) &= U(x^*,y^*) = \alpha\ln(x^*) + \beta\ln(y^*) \\ &= \alpha\ln\left(\frac{\alpha I}{(\alpha+\beta)p}\right) + \beta\ln\left(\frac{\beta I}{(\alpha+\beta)q}\right). \end{aligned}$$

(ii) Differentiate it with respect to I directly:

$$\frac{\partial V(p,q,I)}{\partial I} = \frac{(\alpha+\beta)}{I}.$$

Marginal Utility of Income

- In Slide 9, we have $\lambda = \frac{(\alpha+\beta)}{I}$.

- Therefore, we could have known the utility increment per unit of marginal addition to income, or *Marginal Utility of Income*, without calculating $\frac{\partial V(p,q,I)}{\partial I}$ directly.

Marginal Utility of Income

- Below, we reiterate the argument in Chapter 1.

- First, we write out the problem properly as follows:

$$V(p_1, p_2, I) = \max_{x_1, x_2 \geq 0} U(x_1, x_2)$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 = I.$$

Marginal Utility of Income

The argument proceeds as follows:

(i) Suppose that we have an interior solution, then the consumer would be indifferent between spending the extra amount $\mathrm{d}I$ of income on good 1 or good 2.

- To see this, spending the additional income on good 1 gives additional $MU_1\mathrm{d}I/p_1$ units of utility and spending on good 2 gives additional $MU_2\mathrm{d}I/p_2$ units of utility.

Marginal Utility of Income

- We could show the equivalence of the two utility increments, or MU_1/p_1 and MU_2/p_2, by the first-order necessary conditions.

- The Lagrangian of the problem is

$$\mathcal{L}(x, \lambda) = U(x_1, x_2) + \lambda(I - p_1x_1 - p_2x_2.)$$

- The first-order necessary conditions on x_1 and x_2 suggest $\lambda = MU_1/p_1 = MU_2/p_2$.

Marginal Utility of Income

(ii) Suppose otherwise, that one of the goods attains a corner solution, say $x_2^* = 0$.

Then, by the first-order necessary conditions, we know $\lambda = MU_1/p_1 \geq MU_2/p_2$.

- Therefore, spending $\mathrm{d}I$ on good 1 gives weakly more utility increment, that is, $MU_1\mathrm{d}I/p_1 \geq MU_2\mathrm{d}I/p_2$, and the utility increment is again equal to $\lambda\,\mathrm{d}I$.

Two variables, one constraint

- In the following discussions, we assume that the choice variables attain **interior solutions**, or that we do not impose any non-negativity constraints.

- However, you should keep in mind that the result extends to the situations where the choice variables attain corner solutions (argument (ii)).

Two variables, one constraint

The maximization problem is

$$v = \max_{x_1, x_2} F(x_1, x_2) \tag{MP1}$$

$$\text{s.t. } G(x) = c.$$

Claim. The Lagrange multiplier λ measures how much the highest attainable value v would increase due to a marginal addition to c.

Two variables, one constraint

- Suppose c increases by an infinitesimal amount $\mathrm{d}c$.

- The maximization problem becomes

$$v + \mathrm{d}v = \max_{x_1, x_2} F(x_1, x_2) \qquad \text{(MP2)}$$
$$\text{s.t. } G(x) = c + \mathrm{d}c.$$

- $v + \mathrm{d}v$ represents the new optimum value.

Two variables, one constraint

- We follow notations in the previous chapters and define the solution to (MP1) $x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$.

- We further define $x^* + \mathrm{d}x^* = \begin{pmatrix} x_1^* + \mathrm{d}x_1^* \\ x_2^* + \mathrm{d}x_2^* \end{pmatrix}$ to be the solution to our new maximization problem (MP2).

- Note that $\mathrm{d}x^*$ is not arbitrary; it is the *optimum* small change in the choice, arising in response to a small change in c.

Two variables, one constraint

$$\mathrm{d}v \underbrace{=}_{\text{by definition}} (v + \mathrm{d}v) - v \underbrace{=}_{\text{by definition}} F(x^* + \mathrm{d}x^*) - F(x^*)$$

$$\underbrace{=}_{\text{Taylor approximation}} F_1(x^*)\mathrm{d}x_1^* + F_2(x^*)\mathrm{d}x_2^*$$

$$\underbrace{=}_{\text{First-order conditions}} \lambda G_1(x^*)\mathrm{d}x_1^* + \lambda G_2(x^*)\mathrm{d}x_2^*$$

$$= \lambda \left[G_1(x^*)\mathrm{d}x_1^* + G_2(x^*)\mathrm{d}x_2^*\right]$$

$$\underbrace{=}_{\text{Taylor approximation}} \lambda \left[G(x^* + \mathrm{d}x^*) - G(x^*)\right]$$

$$\underbrace{=}_{\text{constraints}} \lambda \left[(c + \mathrm{d}c) - c\right] = \lambda\, \mathrm{d}c$$

Two variables, one constraint

The result

$$\mathrm{d}v = \lambda \, \mathrm{d}c$$

could be written as follows:

$$\mathrm{d}v/\mathrm{d}c = \lambda. \tag{4.1}$$

Thus, *the Lagrange multiplier is the rate of change of the maximum attainable value of the objective function with respect to a change in the parameter on the right-hand side of the constraint.*

More variables and more constraints

The maximization problem is

$$v = \max_{x_1, x_2, ..., x_n} F(x_1, x_2, ..., x_n) \tag{MP3}$$

$$\text{s.t. } G^1(x) = c_1, G^2(x) = c_2, ..., G^m(x) = c_m.$$

In matrix notation, it is

$$v = \max_{x} F(x) \tag{MP3'}$$

$$\text{s.t. } G(x) = c.$$

More variables and more constraints

- We first consider a change of only one constraint.
- Suppose, say, c_1 increases by an infinitesimal amount $\mathrm{d}c_1$.[1]
- The maximization problem becomes

$$v + \mathrm{d}v = \max_{x_1, x_2, ..., x_n} F(x_1, x_2, ..., x_n) \tag{MP4}$$

$$\text{s.t. } G^1(x) = c_1 + \mathrm{d}c_1, G^2(x) = c_2, ..., G^m(x) = c_m.$$

[1] You will see that the calculation for a change of only one constraint is no simpler than the calculation for changes in many constraints.

More variables and more constraints

- Again, $\boldsymbol{v} + \mathbf{d}\boldsymbol{v}$ represents the new optimum value.
- We denote the solution to (MP3) as $\boldsymbol{x}^*$ and the solution to our new maximization problem (MP4) as $\boldsymbol{x}^* + \mathbf{d}\boldsymbol{x}^*$.

Note that even though only one constraint changes, we need to reoptimize and all x_j^* might change.

$$\mathrm{d}v \underbrace{=}_{\text{by definition}} F(x^* + \mathrm{d}x^*) - F(x^*) \underbrace{=}_{\text{Taylor approximation}} F_1(x^*)\mathrm{d}x_1^* + ... + F_n(x^*)\mathrm{d}x_n^*$$

$$\underbrace{=}_{\substack{\uparrow \\ \text{first-order conditions}}} \sum_{i=1}^{m} \left[\lambda_i G_1^i(x^*)\right] \mathrm{d}x_1^* + ... + \sum_{i=1}^{m} \left[\lambda_i G_n^i(x^*)\right] \mathrm{d}x_n^*$$

$$= \sum_{i=1}^{m} \left[\lambda_i G_1^i(x^*)\mathrm{d}x_1^*\right] + ... + \sum_{i=1}^{m} \left[\lambda_i G_n^i(x^*)\mathrm{d}x_n^*\right]$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} \left[\lambda_i G_j^i(x^*)\mathrm{d}x_j^*\right] = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\lambda_i G_j^i(x^*)\mathrm{d}x_j^*\right]$$

$$= \sum_{i=1}^{m} \left\{\lambda_i \sum_{j=1}^{n} \left[G_j^i(x^*)\mathrm{d}x_j^*\right]\right\} \underbrace{=}_{\substack{\uparrow \\ \text{Taylor approximation}}} \sum_{i=1}^{m} \left\{\lambda_i \left[G^i(x^* + \mathrm{d}x^*) - G^i(x^*)\right]\right\}$$

$$\underbrace{=}_{\substack{\uparrow \\ \text{constraints}}} \sum_{i=1}^{m} \left\{\lambda_i \left[c_i + \mathrm{d}c_i - c_i\right]\right\} = \sum_{i=1}^{m} \lambda_i \mathrm{d}c_i \underbrace{=}_{\text{change in } c_1 \text{ only}} \lambda_1 \, \mathrm{d}c_1.$$

More variables and more constraints

- Therefore, $\mathrm{d}v = \lambda_1 \mathrm{d}c_1$ if we only consider a marginal change in c_1 and remain unchanged all the other constraints.

- Actually, we already obtained the result for simultaneous changes of multiple constraints:

$$\mathrm{d}v = \sum_{i=1}^{m} \lambda_i \, \mathrm{d}c_i.$$

Vector-matrix form

If you are familiar with the vector-matrix notation, the calculation is much simpler:

$$\mathrm{d}v \underbrace{=}_{\text{by definition}} F(x^* + \mathrm{d}x^*) - F(x^*) \underbrace{=}_{\text{Taylor approximation}} F_x(x^*)\mathrm{d}x^*$$

$$\underbrace{=}_{\text{First-order conditions}} \lambda G_x(x^*)\mathrm{d}x^* \underbrace{=}_{\substack{\text{Taylor}\\ \text{approximation}}} \lambda\left[G(x^* + \mathrm{d}x^*) - G(x^*)\right]$$

$$\underbrace{=}_{\text{constraints}} \lambda\left[(c + \mathrm{d}c) - c\right] = \lambda\,\mathrm{d}c$$

More variables and more constraints

Result (Interpretation of Lagrange Multipliers).

If

$$v = \max_x F(x)$$

$$s.t.\ G(x) = c.$$

and λ *is the row vector of multipliers for the constraints, then change* $\mathrm{d}v$ *that results from an infinitesimal change* $\mathrm{d}c$ *is given by*

$$\mathrm{d}v = \lambda\,\mathrm{d}c. \tag{4.3}$$

4.C. Shadow Prices

In the following section, we will explain (4.3):

$$\mathrm{d}v = \lambda \, \mathrm{d}c. \tag{4.3}$$

and discuss the economic meaning of λ.

Marginal Product of Labor

- Consider a planned economy for which a production plan x^* is to be chosen to maximize a social welfare function $F(x)$.

- The vector of the plan's resources requirement is $G(x)$, and the vector of the available amounts of these resources is c.

Marginal Product of Labor

$$v = \max_x \underbrace{F(x)}_{\text{social welfare function}} \tag{MP5}$$

$$\text{s.t.} \quad \underbrace{G(x) = c}_{\text{resource constraints}}.$$

- Assume that the first constraint $G^1(x) = c_1$ is labor constraint.
- Suppose the problem has been solved and the vector of Lagrange multipliers λ is known.

Marginal Product of Labor

- Now, suppose some power outside the economy puts a small additional amount $\mathrm{d}c_1$ of labor into the economy.

- We know from the previous analysis that without further calculation, we already know the resultant increase in social welfare, which is simply $\lambda_1\,\mathrm{d}c_1$.

- We can then say that the Lagrange multiplier λ_1 is the *marginal product of labor* in this economy, measured in units of its social welfare.

Demand Price

- Now suppose that the additional labor can only be used at some cost.

- The maximum the economy is willing to pay in terms of its social welfare units is λ_1 per marginal unit of c_1.

- In this natural sense, λ_1 is the *demand price* the planner places on labor services.

Demand Price

- You may find a price expressed in units of social welfare strange.

- The critic makes sense, however, the more important indicator is the relative demand prices of different resources, rather than the absolute demand prices of single resources.

- The relative demand prices govern the economy's willingness to exchange one resource for another.

Demand Price

- Assume that $G^2(x) = c_2$ is land constraint.
- We are interested to know how much land the economy is willing to give up for an additional $\mathrm{d}c_1$ of labor.
- Assume the amount of land to give up is $\mathrm{d}c_2$.
- Then the net gain in social welfare from this transaction is $\lambda_1 \,\mathrm{d}c_1 - \lambda_2 \,\mathrm{d}c_2$.
- Therefore, the most land the economy is willing to give up is $\lambda_1/\lambda_2 \,\mathrm{d}c_1$.

Demand Price

- The relative demand prices is very relevant to the theory of international trade.

- The simple intuition is that if a neighboring economy has a different trade-off between two resources, then there is a possibility of mutually advantageous trade.[2]

- We will not go deep into this topic.

[2]The trade could be directly on the factors, or indirectly through goods made of these factors.

“Invisible Hand”

- Now, we will discuss the link between market prices and Lagrange multipliers.

- Consider an economy that allocates resources using market.

- In equilibrium, the prices are determined by supplies and demands.

“Invisible Hand”

- Suppose that an economist works out a planner’s problem

$$v = \max_x \underbrace{F(x)}_{\text{social welfare function}} \tag{MP5}$$

$$\text{s.t.} \quad \underbrace{G(x) = c}_{\text{resource constraints}}.$$

and gets a vector of Lagrange multipliers for the resource constraints.

- The social welfare function could be viewed as the criterion to evaluate the performance of the economy.

“Invisible Hand”

Question. Could the market economy replicate the planned allocation, which is the best allocation for a given criterion?

“Invisible Hand”

- There are important cases where the optimum can be replicated in the market.

- Lagrange multipliers are proportional to the market prices of the resources: the relative prices equal the corresponding ratios of multipliers.

- In such cases, the economist would say that the economy is guided by an “*invisible hand*” to his planned optimum. (See Example 4.1)

“Invisible Hand”

To evoke the connection with prices, and yet maintain a conceptual distinction from market prices, Lagrange multipliers are often called *shadow prices*.

4.D. Inequality Constriants

- In economic applications, it is reasonable to consider inequality constraints.

- Full employment of resources may not be optimal.

  (See Example 3.2 *Technological Unemployment*)

- In fact, the study of inequality constraints also turns out to be important in understanding the meaning of Lagrange multipliers λ.

Inequality Constriants

The main problem with equality constraints is

- Because of the connection between prices and *shadow prices* (the Lagrange multipliers), we do expect the Lagrange multipliers to be non-negative.
- However, the maximization problems with equality constraints do not impose any restrictions on the sign.

Inequality Constriants

- The reason is that for equality constraints, an increase in the right-hand side of a constraint equation does not necessarily mean a relaxation of the constraint.

- More specifically, the equality constraint $G^i(x) = c_i$ could be written as $-G^i(x) = -c_i$.

- Such problems could be avoided if we write the constraints as inequality constraints.

Inequality Constriants

The maximization problem with inequality constraints is:

$$v = \max_{x} \underbrace{F(x)}_{\text{social welfare function}} \tag{MP6}$$

$$\text{s.t. } \underbrace{G(x) \leq c}_{\text{resource constraints}}.$$

- For inequality constraints, we invoke Kuhn-Tucker Theorem.

Inequality Constriants

- First-order necessary conditions on x_j's are still valid.

- Therefore, we could repeat our analysis for equality constraints, up until the point where constraints come into play:

$$\mathrm{d}v \underbrace{=}_{\text{by definition}} F(x^* + \mathrm{d}x^*) - F(x^*) \underbrace{=}_{\text{Taylor approximation}} F_x(x^*)\mathrm{d}x^*$$

$$\underbrace{=}_{\text{First-order conditions}} \lambda G_x(x^*)\mathrm{d}x^* \underbrace{=}_{\substack{\text{Taylor}\\ \text{approximation}}} \lambda \left[G(x^* + \mathrm{d}x^*) - G(x^*)\right].$$

Inequality Constriants

- If the constraints are binding for x^* and continue to be binding for $x^* + \mathrm{d}x^*$, we could complete the analysis as we did for the equality constraints:

$$\mathrm{d}v = ... = \lambda \left[G(x^* + \mathrm{d}x^*) - G(x^*)\right] = \lambda \left[(c + \mathrm{d}c) - c\right] = \lambda \, \mathrm{d}c$$

- Whether the constraints are binding is related to the first-order neccessary conditions for λ.

Inequality Constriants

- The first-order necessary conditions for λ give:

$$\mathcal{L}_\lambda(x^*, \lambda) = c - G(x^*) \geq 0,\ \lambda \geq 0,\ \text{with complementary slackness}$$

- The above conditions ensure non-negative Lagrange multipliers λ.

- This is the desirable property that we expect: *shadow prices* λ are non-negative.

Inequality Constriants

Complementary slackness means that, for every i, at least one of the pair

$$G^i(x^*) \leq c_i \text{ and } \lambda_i \geq 0$$

holds with equality. That is,

(i) If resource i is not fully employed ($G^i(x^*) < c_i$), then its shadow price is zero ($\lambda_i = 0$).

(ii) If a resource is with a positive shadow price $\lambda_i > 0$, then it must be fully employed ($G^i(x^*) = c_i$).

Inequality Constriants: Interpretation of shadow prices

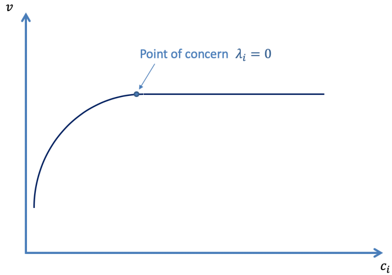
- If part of some resource is already idle, then any increment in it will also be left idle. The maximum value of the objective function will not change, and the shadow price will be zero.

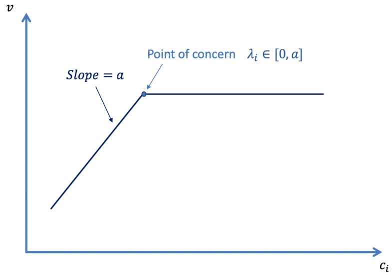
- On the other hand, a positive shadow price means that a marginal increment in resource availability can be put to good use. Then none of the amount originally available can have been left idle in the original plan.

Inequality Constraints: Tricky point

Suppose that c_i is such that

- resource i is fully used ($G^i(x^*) = c_i$),
- but any increment will be left unused.





4.E. Examples

Example 4.1: "Invisible Hand" - Distribution

Consider the stage of planning where the production of the various goods is already known, and the only remaining question is that of distributing them among the consumers. There are C consumers, labeled $c = 1, 2, ..., C$, and G goods, labeled $g = 1, 2, ..., G$. Let X_g be the fixed total amount of good g, and x_{cg} the amoung allocated to consumer c.

Example 4.1: "Invisible Hand" - Distribution (continued)

Each consumer's utility is a function only of his own allocation:

$$u_c = U^c(x_{c1}, x_{c2}, ..., x_{cG}). \tag{4.2}$$

Social welfare is a function of these utility levels:

$$w = W(u_1, u_2, ..., u_C).$$

Assume that the utilities and social welfare function are increasing functions in their respective arguments. Assume also that at the social optimum $x^*_{cg} > 0$ for all c and g.

Example 4.1: "Invisible Hand" - Distribution (continued)

The constraints are

$$x_{1g} + x_{2g} + ... + x_{Cg} \leq X_g, \text{ for } g = 1, 2, ..., G. \qquad (4.3)$$

Question 1: Write down the first-order conditions for the socially optimal allocation.

Question 2: Now suppose the Lagrange multipliers, or *shadow prices*, are made the prices of the goods.

Example 4.1: "Invisible Hand" - Distribution (continued)

Question 2 (continued): Every consumer c is given a money income I_c and allowed to choose his consumption bundle to maximize his utility (4.2) subject to his budget cosntraint. Show that by adjusting money income I_c, the social optimum is attainable. This is when the distribution of income is such that at the margin the social value of every consumer's income is the same. The attainment of the social optimum in the decentralized problem is the "invisible hand" result for the distribution problem.

Example 4.1: Solution

See Lecture Notes.

Example 4.2: Duty-Free Purchases

Imagine the consumption decision of a jet-setter. He can buy various brands of liquor at his home-town store, or at the duty-free stores of the various airports he travels through. The duty-free stores have cheaper prices, but the total quantity he can buy there is restricted by his home country's customs regulations.

Example 4.2: Duty-Free Purchases (continued)

There are n brands. Let p be the row vector of home-town prices and q that of duty-free prices. The duty-free prices are uniformly lower: $q \ll p$. Let x be the column vector of his home-town purchases and y that of the duty-free. Assume that the quantities as continuous variables. Suppose during the year, only K bottles of duty-free liquor is allowed, that is,

$$y_1 + y_2 + ... + y_n \leq K.$$

The jet-setter's total consumption is $c = x + y$, and he derives utility $U(c)$ from liquor consumption.

Example 4.2: Duty-Free Purchases (continued)

Also assume that the income allocated to liquor consumption is fixed at I. Thus, the budget constraint is

$$px + qy \leq I.$$

How much liquor should be jet-setter buy, and from which source?

Example 4.2: Solution

See Lecture Notes.