Chapter 10. Time: The Maximization Principle

As in the case of uncertainty, the optimization over time does not generally require new principles. The variables can be of different dates. At the time the decision is made, there may be uncertainties about the future events. We could deal with such uncertainties using the expected utility formulation as we learned in Chapter 9. The more subtle problem is that there may be opportunities to revise the current decision at some future date. It may be beneficial to allow future revisions as new information may arrive as time goes on. It may also be beneficial to make commitment not to revise since today's preference and the future preferences may not be aligned. We incorporate all these considerations into the objective function and the constraints and the theory of the previous chapters apply. The reason why we study the optimization over time as a separate topic is that such problems have a special structure. Multi-period problems are also common in economic applications.

10.A. An Example of Discrete-Time Optimization

Consider the following life-cycle saving problem. Time is discrete and denoted by $t = 0, 1, 2, ..., T$. For the moment, consider finite T. We will also discuss infinite-period problem where $T = \infty$. A worker gets paid wage w_t in period t. Wage payment can be different from period to period but there is no uncertainty. The decision is on how much of the income to spend on consumption in each period. The unspent income is saved and the overspent income is on debt. The worker receives interest at rate r_t for accumulated savings or pays the accumulated debts at the same rate in period t. Let c_t be the consumption in period t and k_{t+1} be the accumulated savings or debts at the beginning of period $t + 1$. The budget constraint in period t is

$$c_t + k_{t+1} = w_t + (1 + r_t)k_t.$$

$k_0 \geq 0$ is given. Furthermore, at the end of period T, the worker is not allowed to have debts, i.e., $k_{T+1} \geq 0$ is imposed.

The worker only derives utility from consumption and chooses the consumption path to maximize the total present discounted value of utilities in period $t = 0$:

$$U(c_0, c_1, ..., c_T) = \sum_{t=0}^{T} \beta^t u(c_t),$$

where $\beta \in (0, 1)$ is the discount factor. For simplicity, we assume $\lim_{c_t \to 0} u'(c_t) = \infty$ so that $c_t > 0$ for all t.

Example 10.1 (Two-period Case ($T = 1$))**.** The maximization problem for the two-period problem is:

$$\begin{aligned} &\max_{c_0, c_1, k_1, k_2} u(c_0) + \beta u(c_1) \\ \text{s.t. } & c_0 + k_1 = w_0 + (1 + r_0)k_0 \\ & c_1 + k_2 = w_1 + (1 + r_1)k_1 \\ & k_2 \geq 0 \end{aligned}$$

To solve the problem, we use the Kuhn-Tucker theorem as before. Let π_1 and π_2 be the Lagrangian multipliers. The Lagrangian is:

$$\begin{aligned} \mathcal{L}(c_0, c_1, k_1, k_2, \pi_1, \pi_2) =& u(c_0) + \beta u(c_1) + \pi_1[w_0 + (1 + r_0)k_0 - c_0 - k_1] \\ & + \pi_2[w_1 + (1 + r_1)k_1 - c_1 - k_2]. \end{aligned}$$

Kuhn-Tucker conditions:

$$\frac{\partial \mathcal{L}}{\partial c_0} = u'(c_0) - \pi_1 = 0 \tag{10.1}$$

$$\frac{\partial \mathcal{L}}{\partial c_1} = \beta u'(c_1) - \pi_2 = 0 \tag{10.2}$$

$$\frac{\partial \mathcal{L}}{\partial k_1} = -\pi_1 + \pi_2(1 + r_1) = 0 \tag{10.3}$$

$$\frac{\partial \mathcal{L}}{\partial k_2} = -\pi_2 \leq 0, \; k_2 \geq 0 \text{ with complementary slackness} \tag{10.4}$$

$$\frac{\partial \mathcal{L}}{\partial \pi_1} = w_0 + (1 + r_0)k_0 - c_0 - k_1 = 0 \tag{10.5}$$

$$\frac{\partial \mathcal{L}}{\partial \pi_2} = w_1 + (1 + r_1)k_1 - c_1 - k_2 = 0 \tag{10.6}$$

1. CS in (10.4) means $\pi_2 k_2 = 0$: either $k_2 = 0$, or $\pi_2 = 0$ (the shadow value of k_2 is zero).

π_2 could be expressed using (10.2), CS becomes:

$$\beta u'(c_1)k_2 = 0.$$

Since $\beta u'(c_1) > 0$, we must have $k_2 = 0$.

2. (10.1), (10.2) and (10.3) implies the optimal consumption path:

$$\underbrace{u'(c_0)}_{\text{marginal benefit}} = \underbrace{\beta(1+r_1)u'(c_1)}_{\text{marginal cost}}. \tag{10.7}$$

This is the *Euler equation*: an inter-temporal version of first-order condition for a dynamic choice problem that equates the marginal benefit and the marginal cost. In this consumption problem, if the worker consumes 1 marginal unit today (period 0), he gets a marginal utility of $u'(c_0)$ (the marginal benefit of consumption today); if the worker instead saves the unit, he could consume the $(1 + r_1)$ units tomorrow and gets a marginal utility of $u'(c_1)$ for each unit (the marginal cost of consumption today). The optimal consumption path indicates that the marginal benefit equals the marginal cost.

3. c_0, c_1, k_1 can be solved from two constraints (10.5), (10.6) and Euler equation (10.7).

Let us return to the case with arbitrary T. Similar to the two-period case, we could first state the maximization problem and solve the problem using Kuhn-Tucker theorem. The maximization problem is:

$$\max_{\substack{c_0,c_1,\dots,c_T \\ k_1,k_2,\dots,k_{T+1}}} \sum_{t=0}^{T} \beta^t u(c_t)$$

$$\text{s.t. } c_t + k_{t+1} = w_t + (1+r_t)k_t \text{ for all } t = 0, \dots, T$$

$$k_{T+1} \geq 0$$

Let π_{t+1} be the Lagrangian multipliers. The Lagrangian is:

$$\mathcal{L}(c_0, \dots, c_T, k_1, \dots, k_{T+1}, \pi_1, \dots, \pi_{T+1}) = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \pi_{t+1}[w_t + (1+r_t)k_t - c_t - k_{t+1}]$$

Remark 1. Note that for any $t = 0, \dots, T-1$, k_{t+1} appears in two terms: $w_{t+1} + (1 + r_{t+1})k_{t+1} - c_{t+1} - k_{t+2}$ and $w_t + (1 + r_t)k_t - c_t - k_{t+1}$.

We rearrange the expression so that k_t appears in only one term of the sum:

$$\begin{aligned}\mathcal{L} &= \sum_{t=0}^{T}[\beta^t u(c_t) - \pi_{t+1}c_t] + \sum_{t=0}^{T}\pi_{t+1}w_t + \sum_{t=0}^{T-1}\pi_{t+2}(1+r_{t+1})k_{t+1} + \pi_1(1+r_0)k_0 - \sum_{t=1}^{T-1}\pi_{t+1}k_{t+1} - \pi_{T+1}k_{T+1} \\ &= \sum_{t=0}^{T}[\beta^t u(c_t) - \pi_{t+1}c_t] + \sum_{t=0}^{T-1}[\pi_{t+2}(1+r_{t+1}) - \pi_{t+1}]k_{t+1} + \pi_1(1+r_0)k_0 - \pi_{T+1}k_{T+1} + \sum_{t=0}^{T}\pi_{t+1}w_t.\end{aligned}$$

Kuhn-Tucker conditions:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \pi_{t+1} = 0 \text{ for all } t = 0, ..., T \tag{10.8}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = \pi_{t+2}(1+r_{t+1}) - \pi_{t+1} = 0 \text{ for all } t = 0, ..., T-1 \tag{10.9}$$

$$\frac{\partial \mathcal{L}}{\partial k_{T+1}} = -\pi_{T+1} \leq 0, \;\; k_{T+1} \geq 0 \text{ with CS} \tag{10.10}$$

$$\frac{\partial \mathcal{L}}{\partial \pi_{t+1}} = w_t + (1+r_t)k_t - c_t - k_{t+1} = 0 \text{ for all } t = 0, ..., T \tag{10.11}$$

1. CS in (10.10) means $\pi_{T+1}k_{T+1} = 0$. Expressing π_{T+1} using (10.8) gives:

   $$\beta^T u'(c_T)k_{T+1} = 0.$$

   Since $\beta^T u'(c_T) > 0$, we must have $k_{T+1} = 0$.

2. (10.8) and (10.9) implies the optimal consumption path:

   $$\underbrace{u'(c_t)}_{\text{marginal benefit}} = \underbrace{\beta(1+r_{t+1})u'(c_{t+1})}_{\text{marginal cost}} \text{ for all } t = 0, ..., T-1 \tag{10.12}$$

   This is again the *Euler equations*.

3. c_t for $t = 0, ..., T$ and k_{t+1} for $t = 0, ..., T-1$ could be solved from the constraints (10.11) and the Euler equations (10.12).

Consumption Pattern.

1. Suppose $\beta(1+r_{t+1}) = 1$ for all $t = 0, ..., T-1$. Then according to (10.12), $u'(c_t) = u'(c_{t+1})$. If $u(c)$ is strictly concave, then $c_t = c_{t+1}$, i.e., consumption does not vary over time.

2. Suppose $\beta(1+r_{t+1}) > 1$, i.e., $\beta > \frac{1}{1+r_{t+1}}$ for all $t = 0, ..., T-1$. Then according to (10.12), $u'(c_t) > u'(c_{t+1})$. If $u(c)$ is strictly concave, then $c_t < c_{t+1}$. That is, when the worker is patient (β large), he is willing to save more for future consumption.

3. Suppose $\beta(1+r_{t+1}) < 1$, i.e., $\beta < \frac{1}{1+r_{t+1}}$ for all $t = 0, ..., T-1$. Then according to (10.12), $u'(c_t) < u'(c_{t+1})$. If $u(c)$ is strictly concave, then $c_t > c_{t+1}$. That is, when the worker is impatient (β small), he is willing to consume more today.

10.B. General Problem of Discrete-Time Optimization

10.B.1. Statement of the Problem

The most important aspect of the optimization over time is the stock-flow relationships. A *stock* is measured at one specific time, and represents a quantity existing at that point in time. A *flow* is measured over an interval of time. Denote y_t as the *stock* variables and z_t as the *flow* variables. In mathematical terminology, stocks are called *state variables* and flows are called *control variables*. In the previous example, k_t are the state (or stock) variables and c_t are the control (or flow) variables.

The increment of stocks depends on both the stocks and the flows during that period. The production possibility constraints are

$$y_{t+1} - y_t = Q(y_t, z_t, t). \tag{10.13}$$

Q should be thought of as a production function. t as an argument of Q captures exogenous technological change. Mathematically, the control variables govern the change in the state variables. In the previous example, we have $k_{t+1} - k_t = w_t + r_t k_t - c_t$.

In addition to the above constraints that govern the changes in stocks, there may be constraints on all variables pertaining to any one date,

$$G(y_t, z_t, t) \leq 0, \tag{10.14}$$

where G is a vector function. In the previous example, we need $c_t \geq 0$ for all t. (Even though we ignore such constraints in the calculations by assuming $\lim_{c_t \to 0} u'(c_t) = \infty$ so that $c_t > 0$ for all t.)

It is assumed that the criterion function is *additively separable*:

$$\sum_{t=0}^{T} F(y_t, z_t, t). \tag{10.15}$$

We take the initial stock y_0 as given. For simplicity, we also take the terminal value y_{T+1} to be fixed. Note that in the previous example, we instead have the terminal condition $k_{T+1} \geq 0$. The cases with $y_{T+1} \geq 0$, or more generally $y_{T+1} \geq \hat{y}$ are discussed at the end of this section (Transversality condition).

10.B.2. The Maximum Principle

The choice variables are y_t for $t = 1, 2, ...T$ and z_t for $t = 0, 1, 2, ...T$.

Let λ_t denote the multipliers for the constraints (10.14) and π_{t+1} denote the multipliers for the constraints (10.13). In terms of economic interpretations, λ_t are the usual shadow prices of the constraints (10.14); π_{t+1} are the shadow prices of y_{t+1}. **The Lagrangian of the full inter-temporal problem** is

$$\mathcal{L} = \sum_{t=0}^{T} \left\{ F(y_t, z_t, t) + \pi_{t+1} \left[y_t + Q(y_t, z_t, t) - y_{t+1} \right] - \lambda_t G(y_t, z_t, t) \right\} \tag{10.16}$$

The arguments in $\mathcal{L}$ are all $y_t, z_t, \lambda_t, \pi_{t+1}$.

1. FOC with respect to z_t for $t = 0, 1, ..., T$ are

$$\partial \mathcal{L} / \partial z_t \equiv F_z(y_t, z_t, t) + \pi_{t+1} Q_z(y_t, z_t, t) - \lambda_t G_z(y_t, z_t, t) = 0. \tag{10.17}$$

2. FOC with respect to y_t is more complicated since y_t appears in two terms of the sum. We first simplify the term $\sum_{t=0}^{T} \pi_{t+1}(y_t - y_{t+1})$.

$$\begin{aligned}
\sum_{t=0}^{T} \pi_{t+1}(y_t - y_{t+1}) &= \pi_1(y_0 - y_1) + \pi_2(y_1 - y_2) + ... + \pi_{T+1}(y_T - y_{T+1}) \\
&= y_0\pi_1 + y_1(\pi_2 - \pi_1) + ... + y_T(\pi_{T+1} - \pi_T) - y_{T+1}\pi_{T+1} \\
&= \sum_{t=1}^{T} y_t(\pi_{t+1} - \pi_t) + y_0\pi_1 - y_{T+1}\pi_{T+1}
\end{aligned}$$

Then (10.16) becomes

$$\begin{aligned}
\mathcal{L} = &\sum_{t=1}^{T} \left\{ F(y_t, z_t, t) + \pi_{t+1} Q(y_t, z_t, t) + y_t(\pi_{t+1} - \pi_t) - \lambda_t G(y_t, z_t, t) \right\} \\
&+ F(y_0, z_0, 0) + \pi_1 Q(y_0, z_0, 0) + y_0\pi_1 - y_{T+1}\pi_{T+1} - \pi_1 G(y_0, z_0, 0)
\end{aligned} \tag{10.18}$$

FOC with respect to y_t for $t = 1, ..., T$ (note that y_0 and y_{T+1} are not included) are

$$\partial\mathcal{L}/\partial y_t \equiv F_y(y_t, z_t, t) + \pi_{t+1}Q_y(y_t, z_t, t) + \pi_{y+1} - \pi_t - \lambda_t G_y(y_t, z_t, t) = 0,$$

$$\text{or } \pi_{t+1} - \pi_t = -\left[F_y(y_t, z_t, t) + \pi_{t+1}Q_y(y_t, z_t, t) - \lambda_t G_y(y_t, z_t, t)\right]. \quad (10.19)$$

3. FOC with respect to π_{t+1} for $t = 0, ..., T$ are

$$\partial\mathcal{L}/\partial\pi_{t+1} = y_t + Q(y_t, z_t, t) - y_{t+1} = 0.$$

   These are the constraints (10.13).

4. FOC with respect to λ_t for $t = 0, ..., T$ are

$$G(y_t, z_t, t) \le 0,\ \ \lambda_t \ge 0 \text{ with CS.} \quad (10.20)$$

Hamiltonian. The conditions could be written in a more compact and economic illuminating way. Define **Hamiltonian**:

$$H(y_t, z_t, \pi_{t+1}, t) = F(y_t, z_t, t) + \pi_{t+1}Q(y_t, z_t, t). \quad (10.21)$$

The initial inter-temporal maximization problem could be rewritten as T single-period maximization problems:

$$\max_{z_t} H(y_t, z_t, \pi_{t+1}, t) \equiv \max_{z_t} F(y_t, z_t, t) + \pi_{t+1}Q(y_t, z_t, t)$$
$$\text{s.t. } G(y_t, z_t, t) \le 0.$$

In these single-period problems, only z_t are choice variables; y_t, π_{t+1} are parameters. Define **Lagrangian** of the single-period maximization problem:

$$L(z_t, \lambda_t, y_t, \pi_{t+1}, t) = H(y_t, z_t, \pi_{t+1}, t) - \lambda_t G(y_t, z_t, t) \quad (10.22)$$

(10.17) and (10.20) are the conditions for the single-period maximization problem. Write $H^*(y_t, \pi_{t+1}, t)$ the resulting maximum value.

Envelope Theorem applies to the parameters y_t and π_{t+1}:

$$H_y^*(y_t, \pi_{t+1}, t) = L_y(z_t^*, \lambda_t^*, y_t, \pi_{t+1}, t)$$

$$H_\pi^*(y_t, \pi_{t+1}, t) = L_\pi(z_t^*, \lambda_t^*, y_t, \pi_{t+1}, t) = Q(y_t, z_t^*, t)$$

Then (10.19) and (10.13) are more simply written as

$$\pi_{t+1} - \pi_t = -H_y^*(y_t, \pi_{t+1}, t) \tag{10.23}$$

$$y_{t+1} - y_t = H_\pi^*(y_t, \pi_{t+1}, t) \tag{10.24}$$

Theorem 10.1 (The Maximization Principle)**.** *The first-order necessary conditions for the maximization problem*

$$\max_{\{y_t\}_{t=1}^T, \{z_t\}_{t=0}^T} \sum_{t=0}^{T} F(y_t, z_t, t) \tag{10.15}$$

$$s.t. \;\; y_{t+1} - y_t = Q(y_t, z_t, t) \tag{10.13}$$

$$G(y_t, z_t, t) \le 0 \tag{10.14}$$

are

(i) for each t, z_t maximizes the Hamiltonian $H(y_t, z_t, \pi_{t+1}, t)$ subject to the single-period constraints $G(y_t, z_t, t) \le 0$, and

(ii) the changes in y_t and π_t over time are governed by the difference equations (10.23) and (10.24).

Interpretations.

1. For condition (i), it is clear that we would not want to choose z_t to maximize $F(y_t, z_t, t)$: the choice of z_t affects y_{t+1} via (10.13) and therefore affects the terms in the objective function at time $t + 1$ onwards. The effect of z_t on y_{t+1} equals its effect on $Q(y_t, z_t, t)$, and the resulting change in the objective function is the shadow price π_{t+1} of y_{t+1} times $Q(y_t, z_t, t)$. Thus, Hamiltonian offers a simple way to take into account the future consequences of the choices of the controls z_t at t.
2. For condition (ii), (10.23) could be viewed as an **inter-temporal no-arbitrage condition**. A marginal unit of y_t yields the marginal return $F_y(y_t, z_t, t) - \lambda_t G_y(y_t, z_t, t)$

within period t, and an extra $Q_y(y_t, z_t, t)$ the next period evaluated at π_{t+1}. These can be thought of as a dividend. In addition, there is capital gain of $\pi_{t+1} - \pi_t$. When y_t is optimum, the overall marginal return, or the sum of these components, should be 0:

$$\Big(F_y(y_t, z_t, t) - \lambda_t G_y(y_t, z_t, t)\Big) + \pi_{t+1} Q_y(y_t, z_t, t) + \Big(\pi_{t+1} - \pi_t\Big) = 0.$$

Transversality condition. If we change the terminal condition to $y_{T+1} \geq 0$. Then by Kuhn-Tucker condition for y_{T+1}, we need

$$\pi_{T+1} \geq 0 \text{ and } \pi_{T+1} y_{T+1} = 0.$$

That is, if any positive stocks are left, they must be worthless.

More generally, if there is a constraint $y_{T+1} \geq \hat{y}$, then we require

$$\pi_{T+1} \geq 0 \text{ and } \pi_{T+1}(y_{T+1} - \hat{y}) = 0.$$

Such conditions on terminal stocks and the respective shadow prices are called *transversality conditions.*

10.C. Continuous-Time Model

We have treated time as discrete succession of periods. Such a formulation allows us to develop the theory using the technics of Kuhn-Tucker theorem. However, in practice, it is sometimes more convenient to treat time as a continuous variable.

To formulate the problem into a continuous-time problem, we can think of continuous-time models as the limit of discrete-time models when we take the discrete periods of length Δt to 0. Flows are now rates per time. In particular, the constraint (10.13) becomes

$$y(t + \Delta t) - y(t) = Q(y(t), z(t), t)\Delta t.$$

Dividing by Δt and taking the limit $\Delta \to 0$ gives

$$\mathrm{d}y(t)/\mathrm{d}t = Q(y(t), z(t), t).$$

We use $\dot{y}(t)$ to denote the derivative, i.e., $\dot{y}(t) = \mathrm{d}y(t)/\mathrm{d}t$. The objective function (10.15) is modified to be an integral. The problem becomes[1]

$$\max_{y(t),z(t)} \int_0^T F(y(t), z(t), t)\mathrm{d}t \tag{10.15'}$$

$$\text{s.t.}\ \ \dot{y}(t) = Q(y(t), z(t), t) \tag{10.13'}$$

$$G(y(t), z(t), t) \leq 0 \tag{10.14'}$$

The initial condition $y(0)$ and the terminal condition $y(T)$ are given.

Maximum Principle. We also have similar results as what we have derived in the discrete-time model. Define **Hamiltonian**:

$$H(y(t), z(t), \pi(t), t) = F(y(t), z(t), t) + \pi(t)Q(y(t), z(t), t).$$

$\pi(t)$ is called the *co-state variable*. The **Lagrangian** is:

$$L(z(t), \lambda(t), y(t), \pi(t), t) = H(y(t), z(t), \pi(t), t) - \lambda(t)G(y(t), z(t), t).$$

Theorem 10.2. *The first-order necessary conditions for the continuous-time problem is*

(i) $z(t)$ maximizes the Hamiltonian $H(y(t), z(t), \pi(t), t)$ subject to the single period constraints $G(y(t), z(t), t) \leq 0$, and

(ii) $y(t)$ and $\pi(t)$ are governed by the differential equations

$$\dot{\pi}(t) = -H^*_y(y(t), \pi(t), t) \tag{10.23'}$$

$$\dot{y}(t) = H^*_\pi(y(t), \pi(t), t) \tag{10.24'}$$

10.D. Further Discussions

10.D.1. Infinite Horizon problems

There is no last period in the infinite horizon problems. So, unlike the finite-horizon problems, now it is unreasonable to impose non-negative stock in the last period. For

[1] We have used subscripts to denote time for the discrete-time models and we write time as a function argument in the parentheses in the continuous-time models.

the problem to be well-defined, we need to impose the transversality condition. Heuristically, the transversality conditions for infinite-horizon problems listed below are natural extensions of the previous transversality conditions for finite-horizon problems with non-negative terminal stock (which is just the complementary slackness condition).

For discrete-time problems, the transversality condition is:

$$\lim_{T\to\infty} \pi_T y_T = 0.$$

For continuous-time problems, the transversality condition is:

$$\lim_{T\to\infty} \pi(T)y(T) = 0.$$

10.D.2. Present Value v.s. Current Value Hamiltonian

In economic applications, we usually need to discount future values. In the previous example, the worker discount future utilities by β per period. In this section, we analyze the optimization problem with the discount factors explicitly expressed in the objective function. For simplicity, we assume that T is finite, and y_0, y_{T+1} are given. Transversality conditions are needed if T is infinite or we only impose $y_{T+1} \geq 0$ for finite T.

Discrete-Time Model. In order to explicitly taking into account the discount factors β, we rewrite the criterion function (10.15) as:

$$\sum_{t=0}^{T} \beta^t f(y_t, z_t, t). \tag{10.25}$$

Present value Hamiltonian. Similar to the Hamiltonian (10.21), we define the Hamiltonian, called the *present value Hamiltonian*:

$$H^p(y_t, z_t, \pi_{t+1}, t) = \beta^t f(y_t, z_t, t) + \pi_{t+1} Q(y_t, z_t, t).$$

π is the present value multiplier. The Lagrangian is

$$L^p(z_t, \lambda_t, y_t, \pi_{t+1}, t) = H^p(y_t, z_t, \pi_{t+1}, t) - \lambda_t G(y_t, z_t, t)$$

FOCs are:

$$L^p_z(z_t, \lambda_t, y_t, \pi_{t+1}, t) = 0$$

$$\pi_{t+1} - \pi_t = -H^{p^*}_y(y_t, \pi_{t+1}, t)$$

$$y_{t+1} - y_t = H^{p^*}_\pi(y_t, \pi_{t+1}, t) \text{ (the inter-temporal constraint)}$$

$$G(y_t, z_t, t) \le 0,\ \lambda_t \ge 0 \text{ with CS.}$$

Current value Hamiltonian. It is also possible to define Hamiltonian in terms of the current value: $H^c(y_t, z_t, \mu_{t+1}, t) = f(y_t, z_t, t) + \beta\mu_{t+1}Q(y_t, z_t, t).$

This is called the *current value Hamiltonian* and μ is the current value multiplier. Note that $H^p = \beta^t H^c$ and $\pi_t = \beta^t \mu_t$. The Lagrangian is

$$L^c(z_t, \nu_t, y_t, \mu_{t+1}, t) = H^c(y_t, z_t, \pi_{t+1}, t) - \nu_t G(y_t, z_t, t).$$

Note that $L^p = \beta^t L^c$ and $\lambda_t = \beta^t \nu_t$. From the FOCs for the present value Hamiltonian, we could deduce FOCs for the current value Hamiltonian:

$$L^c_z(z_t, \nu_t, y_t, \mu_{t+1}, t) = 0$$

$$\beta\mu_{t+1} - \mu_t = -H^{c^*}_y(y_t, \mu_{t+1}, t)$$

$$\beta(y_{t+1} - y_t) = H^{c^*}_\mu(y_t, \mu_{t+1}, t) \text{ (the inter-temporal constraint)}$$

$$G(y_t, z_t, t) \le 0,\ \nu_t \ge 0 \text{ with CS.}$$

Continuous-Time Model. Discount factor changes from $\beta \equiv \frac{1}{1+\rho}$ to $e^{-\rho t}$ where ρ is the discount rate:

$$\lim_{n\to\infty} \left(\frac{1}{1+\frac{\rho}{n}}\right)^n = \lim_{n\to\infty} \left[\left(1+\frac{\rho}{n}\right)^{\frac{n}{\rho}}\right]^{-\rho} = e^{-\rho}.$$

Thus, we rewrite the criterion function as:

$$\int_0^T e^{-\rho t} f(y(t), z(t), t)\mathrm{d}t.$$

Present value Hamiltonian. We define the *present value Hamiltonian*:

$$H^p(y(t), z(t), \pi(t), t) = e^{-\rho t} f(y(t), z(t), t) + \pi(t) Q(y(t), z(t), t).$$

π is the present value multiplier.

The Lagrangian is

$$L^p(z(t), \lambda(t), y(t), \pi(t), t) = H^p(y(t), z(t), \pi(t), t) - \lambda(t)G(y(t), z(t), t).$$

FOCs are:

$$L^p_z(z(t), \lambda(t), y(t), \pi(t), t) = 0$$
$$\dot{\pi}(t) = -H^{p^*}_y(y(t), \pi(t), t)$$
$$\dot{y}(t) = H^{p^*}_\pi(y(t), \pi(t), t)$$
$$G(y(t), z(t), t) \leq 0,\ \lambda(t) \geq 0 \text{ with CS.}$$

Current value Hamiltonian. It is also possible to define Hamiltonian in terms of the current value:

$$H^c(y(t), z(t), \pi(t), t) = f(y(t), z(t), t) + \mu(t)Q(y(t), z(t), t).$$

This is called the *current value Hamiltonian* and μ is the current value multiplier. Note that $H^p = e^{-\rho t}H^c$ and $\pi(t) = e^{-\rho t}\mu(t)$. The Lagrangian is

$$L^c(z(t), \nu(t), y(t), \pi(t), t) = H^c(y(t), z(t), \pi(t), t) - \nu(t)G(y(t), z(t), t).$$

Note that $L^p = e^{-\rho t}L^c$ and $\lambda(t) = e^{-\rho t}\nu(t)$.

From the FOCs for the present value Hamiltonian, we could deduce FOCs for the current value Hamiltonian:

$$L^c_z(z(t), \nu(t), y(t), \pi(t), t) = 0$$
$$\dot{\mu}(t) - \rho\mu(t) = -H^{c^*}_y(y(t), \mu(t), t)$$
$$\dot{y}(t) = H^{c^*}_\mu(y(t), \mu(t), t)$$
$$G(y(t), z(t), t) \leq 0,\ \lambda_t \geq 0 \text{ with CS.}$$

10.E. Examples

Example 1: Life-Cycle Saving Consider the continuous-time version of the life-cycle saving model. The evolution of k is governed by

$$\dot{k} = w + rk - c.$$

w is the constant wage rate and r is the constant interest rate. Assume that there are no inheritances or bequests:

$$k(0) = k(T) = 0.$$

The instantaneous utility function is $\ln(c)$, and the discount rate is ρ, so the objective function is:

$$\int_0^T e^{-\rho t} \ln(c) \mathrm{d}t.$$

Maximum Principle. We use the maximum principle to solve the problem. Define Hamiltonian:

$$H = e^{-\rho t} \ln(c) + \pi(w + rk - c).$$

The condition for c is:

$$H_c = e^{-\rho t} c^{-1} - \pi = 0 \implies c^* = e^{-\rho t} \pi^{-1} \tag{10.26}$$

Substituting into H:

$$H^* = e^{-\rho t}[-\rho t - \ln(\pi)] + \pi(w + rk) - e^{-\rho t}.$$

Then the differential equations for k and π are:

$$\dot{\pi} = -H_k^* = \pi r \tag{10.27}$$

$$\dot{k} = H_\pi^* = w + rk - e^{-\rho t} \pi^{-1} \tag{10.28}$$

Analysis. The general solution of (10.27) is

$$\pi = \pi_0 e^{-rt}, \tag{10.29}$$

where π_0 is a constant to be determined. Substituting this into (10.28), we have

$$\dot{k} = w + rk - \pi_0^{-1} e^{(r-\rho)t} \tag{10.30}$$

This is a first order differential equation. (10.30) is equivalent to

$$\frac{\mathrm{d}ke^{-rt}}{\mathrm{d}t} = we^{-rt} - \pi_0^{-1} e^{-\rho t},$$

which integrate into

$$ke^{-rt} - k(0) = \frac{w(1 - e^{-rt})}{r} - \frac{\pi_0^{-1}(1 - e^{-\rho t})}{\rho} \tag{10.31}$$

Plugging in $k(0) = k(T) = 0$ gives π_0. This completes the solution: k is given by (10.31), and using (10.26) and (10.29), c is given by

$$c = e^{(r-\rho)t} \pi_0^{-1}. \tag{10.32}$$

Some economic implications from (10.32) (without solving for π_0):

- If $r > \rho$, the worker's optimum consumption grows over time. Since consumption and wages have the same discounted present values, $c < w$ in the early years and $c > w$ in the later years.
- If $r < \rho$, then the worker's optimum consumption decreases over time. So the worker borrows in the early years.

Example 2: Optimum Growth Consider the optimal saving problem from the view of the economy as a whole. The rate of return now is endogenous. Besides, consider $T = \infty$. k denotes the stock of capital. Let $F(k)$ be the production function. F is increasing, strictly concave with $F(0) = 0$ and $F'(0) = \infty$. Capital depreciates at a constant rate δ. c is the consumption flow. Then the capital accumulation equation is

$$\dot{k} = F(k) - \delta k - c. \tag{10.33}$$

The initial capital stock $k(0)$ is given.

The objective is still to maximize the present discounted value of utilities:

$$\int_0^\infty e^{-\rho t}U(c)\mathrm{d}t,$$

where the utility of the flow of consumption $U(c)$ is increasing and strictly concave.

Maximum Principle. To apply the maximum principle, define Hamiltonian:

$$H = e^{-\rho t}U(c) + \pi(F(k) - \delta k - c).$$

The condition for c is:

$$H_c = e^{-\rho t}U'(c) - \pi = 0 \implies e^{-\rho t}U'(c) = \pi. \tag{10.34}$$

The condition for k is:

$$\dot{\pi} = -H_k^* = -\pi(F'(k) - \delta) \tag{10.35}$$

The condition for π gives the capital accumulation equation (10.33) at the optimal c. Furthermore, this is an infinite-horizon problem, so we require the transversality condition:

$$\lim_{T\to\infty} \underbrace{\pi(T)}_{\text{discounted shadow price}} k(T) = 0 \implies \lim_{T\to\infty} e^{-\rho T}U'(c(T))k(T) = 0. \tag{10.36}$$

The transversality condition means that the present value of capital stock in the infinite future is zero.

Analysis. We would like to work with k and c. So we first eliminate π in the condition for c. Differentiation of (10.34) gives

$$\Big(U''(c)\dot{c} - \rho U'(c)\Big)e^{-\rho t} = \dot{\pi} \tag{10.37}$$

Plugging (10.34) and (10.37) into (10.35) and simplifying, we obtain the *Euler equation*:

$$-\frac{U''(c)\dot{c}}{U'(c)} = \Big(F'(k) - (\rho + \delta)\Big) \implies \frac{\dot{c}}{c} = \frac{F'(k) - (\rho + \delta)}{\eta(c)} \tag{10.38}$$

where $\eta(c) = -\frac{cU''(c)}{U'(c)}$ is the elasticity of marginal utility of consumption. $\eta(c) > 0$ since U is increasing and strictly concave.

Now we have (10.33) and (10.38) as the pair of differential equations in k and c. We can show the solutions in the phase diagram. See Figure 10.1.

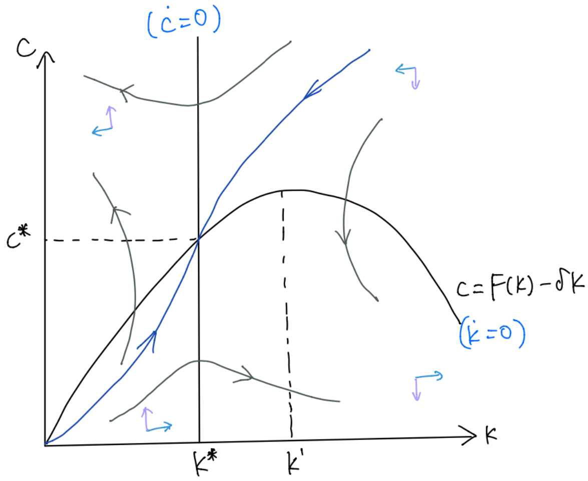

Figure 10.1: Phase diagram for optimum growth

To draw the graph,

1. Identify the two curves $\dot{k} = 0$ and $\dot{c} = 0$. The two curves split the space into four regions.
2. Identify how k and c changes within each region:
   - k increases if $c < F(k) - \delta k$;
   - c increases if $F'(k) > \rho + \delta$.
3. Draw representative paths with different initial points. The paths cannot cross, since the direction of motion is uniquely determined by the equations.

There are two paths converging to the equilibrium (k^*, c^*), one from the left and one from the right. Given $k(0)$, then $c(0)$ should be chosen such that the path starting at $(k(0), c(0))$ converges to (k^*, c^*). Transversality condition (10.36) is satisfied.