Chapter 10. Time:

The Maximization Principle

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Introduction

- Optimization over time does not generally require new principles.
- Variables can be of different dates.
- There may be uncertainties about future events or opportunities to revise current decision at some future date.
- We incorporate all these into objective function and constraints and theory of previous chapters apply.

10.A. Discrete-Time Optimization: Example

Life-cycle saving problem

- Time is discrete and denoted by $t = 0, 1, 2, ..., T$ (T finite)
- Worker gets paid wage w_t in period t.
- Decision is on how much of income to spend on consumption in each period.
  - Unspent income saved; overspent income on debt.
  - Interest rate r_t

Life-cycle saving problem

- c_t: consumption in period t
- k_{t+1}: accumulated savings (or debts), beginning of $t+1$.
- Budget constraint in period t is

$$c_t + k_{t+1} = w_t + (1 + r_t)k_t.$$

  $k_0 \geq 0$ is given.

- Terminal condition: $k_{T+1} \geq 0$.

Life-cycle saving problem

- Objective function:

$$U(c_0, c_1, ..., c_T) = \sum_{t=0}^{T} \beta^t u(c_t),$$

  where $\beta \in (0, 1)$ is the discount factor.

- For simplicity, we assume $\lim_{c_t \to 0} u'(c_t) = \infty$ so that $c_t > 0$ for all t.

Example (Two-period Case ($T = 1$)).

Maximization problem is:

$$\max_{c_0, c_1, k_1, k_2} u(c_0) + \beta u(c_1)$$

$$\text{s.t. } c_0 + k_1 = w_0 + (1 + r_0)k_0$$

$$c_1 + k_2 = w_1 + (1 + r_1)k_1$$

$$k_2 \geq 0$$

Two-period Case ($T = 1$)

- To solve the problem, we use Kuhn-Tucker theorem.
- Let π_1 and π_2 be Lagrangian multipliers.
- Lagrangian is:

$$\begin{aligned}\mathcal{L}(c_0, c_1, k_1, k_2, \pi_1, \pi_2) =& u(c_0) + \beta u(c_1) \\ & + \pi_1[w_0 + (1 + r_0)k_0 - c_0 - k_1] \\ & + \pi_2[w_1 + (1 + r_1)k_1 - c_1 - k_2].\end{aligned}$$

Two-period Case ($T = 1$)

Kuhn-Tucker conditions:

$$\frac{\partial \mathcal{L}}{\partial c_0} = u'(c_0) - \pi_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial c_1} = \beta u'(c_1) - \pi_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial k_1} = -\pi_1 + \pi_2(1 + r_1) = 0$$
$$\frac{\partial \mathcal{L}}{\partial k_2} = -\pi_2 \leq 0,\ k_2 \geq 0 \text{ with CS}$$
$$\frac{\partial \mathcal{L}}{\partial \pi_1} = w_0 + (1 + r_0)k_0 - c_0 - k_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \pi_2} = w_1 + (1 + r_1)k_1 - c_1 - k_2 = 0$$

Two-period Case ($T = 1$)

1. CS and $\pi_2 = \beta u'(c_1) \implies k_2 = 0$.

2. Euler Equation:

$$\underbrace{u'(c_0)}_{\text{marginal benefit}} = \underbrace{\beta(1 + r_1)u'(c_1)}_{\text{marginal cost}}.$$

3. c_1, c_2, k_1 could be solved from two constraints and Euler equation .

Life-cycle saving problem (arbitrary T)

- Similar to two-period case, we could first state maximization problem and solve it using Kuhn-Tucker theorem.
- Maximization problem is:

$$\max_{\substack{c_0, c_1, ..., c_T \\ k_1, k_2, ..., k_{T+1}}} \sum_{t=0}^{T} \beta^t u(c_t)$$

$$\text{s.t. } c_t + k_{t+1} = w_t + (1 + r_t)k_t \text{ for all } t = 0, ..., T$$

$$k_{T+1} \geq 0$$

Life-cycle saving problem

- Let π_{t+1} be Lagrangian multipliers.
- Lagrangian is:

$$\mathcal{L}(c_0, ..., c_T, k_1, ..., k_{T+1}, \pi_1, ..., \pi_{t+1})$$
$$= \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \pi_{t+1}[w_t + (1 + r_t)k_t - c_t - k_{t+1}]$$

Remark 1. Note that for any $t = 0, ..., T - 1$, k_{t+1} appears in two terms: $w_{t+1} + (1 + r_{t+1})k_{t+1} - c_{t+1} - k_{t+2}$ and $w_t + (1 + r_t)k_t - c_t - k_{t+1}$.

Life-cycle saving problem

We rearrange the expression so that k_t appears in only one term of the sum:

$$\mathcal{L} = \sum_{t=0}^{T}[\beta^t u(c_t) - \pi_{t+1}c_t] + \sum_{t=0}^{T-1}[\pi_{t+2}(1 + r_{t+1}) - \pi_{t+1}]k_{t+1}$$

$$+ \pi_1(1 + r_0)k_0 - \pi_{T+1}k_{T+1} + \sum_{t=0}^{T}\pi_{t+1}w_t.$$

Life-cycle saving problem

Kuhn-Tucker conditions:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \pi_{t+1} = 0 \text{ for all } t = 0, ..., T$$
$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = \pi_{t+2}(1 + r_{t+1}) - \pi_{t+1} = 0 \text{ for all } t = 0, ..., T-1$$
$$\frac{\partial \mathcal{L}}{\partial k_{T+1}} = -\pi_{T+1} \leq 0, \;\; k_{T+1} \geq 0 \text{ with CS}$$
$$\frac{\partial \mathcal{L}}{\partial \pi_{t+1}} = w_t + (1 + r_t)k_t - c_t - k_{t+1} = 0 \text{ for all } t = 0, ..., T$$

Life-cycle saving problem

1. CS and $\pi_{T+1} = \beta^T u'(c_T) \implies k_{T+1} = 0$.

2. Euler Equations:

$$\underbrace{u'(c_t)}_{\text{marginal benefit}} = \underbrace{\beta(1 + r_{t+1})u'(c_{t+1})}_{\text{marginal cost}} \text{ for all } t = 0, ..., T - 1$$

3. c_t for $t = 0, ..., T$ and k_{t+1} for $t = 0, ..., T - 1$ could be solved from constraints and Euler equations.

Consumption Pattern

1. Suppose $\beta(1 + r_{t+1}) = 1$ for all $t = 0, ..., T - 1$.
   - Euler equation implies $u'(c_t) = u'(c_{t+1})$.
   - If $u(c)$ is strictly concave, then $c_t = c_{t+1}$ (consumption does not vary over time).
2. Suppose $\beta(1 + r_{t+1}) > 1$ for all $t = 0, ..., T - 1$.
   - Euler equation implies $u'(c_t) > u'(c_{t+1})$.
   - If $u(c)$ is strictly concave, then $c_t < c_{t+1}$.
   - When the worker is patient (β large), he is willing to save more for future consumption.

Consumption Pattern

3. Suppose $\beta(1 + r_{t+1}) < 1$ for all $t = 0, ..., T - 1$.

    - Euler equation implies $u'(c_t) < u'(c_{t+1})$.
    - If $u(c)$ is strictly concave, then $c_t > c_{t+1}$.
    - When the worker is impatient (β small), he is willing to consume more today.

10.B. Discrete-Time Optimization: General

10.B.1. Statement of the Problem

- Most important aspect of optimization over time is stock-flow relationships.
- A stock is measured at one specific time, and represents a quantity existing at that point in time.
- A flow is measured over an interval of time.
- Denote y_t as stock variables and z_t as flow variables.

Statement of the Problem

- In mathematical terminology,
  - stocks are called state variables and
  - flows are called control variables.
- In the previous example,
  - k_t are state (or stock) variables and
  - c_t are control (or flow) variables.

Statement of the Problem

- Increment of stocks depends on both stocks and flows during that period.
- Production possibility constraints are

$$y_{t+1} - y_t = Q(y_t, z_t, t).$$

  - Q should be thought of as a production function.
  - t as an argument of Q: exogenous technological change.
- In previous example, we have $k_{t+1} - k_t = w_t + r_t k_t - c_t$.

Statement of the Problem

- In addition, there may be constraints on all variables pertaining to any one date,

$$G(y_t, z_t, t) \leq 0,$$

where G is a vector function.

- In previous example, we need $c_t \geq 0$ for all t.
  - Even though we ignore such constraints in the calculations by assuming $\lim_{c_t \to 0} u'(c_t) = \infty$ so that $c_t > 0$ for all t.

Statement of the Problem

It is assumed that criterion function is additively separable:

$$\sum_{t=0}^{T} F(y_t, z_t, t).$$

- We take initial stock y_0 as given.
- For simplicity, we take terminal value y_{T+1} to be fixed.
- Note that in previous example, we instead have terminal condition $k_{T+1} \geq 0$.
- Cases with $y_{T+1} \geq 0$, or more generally $y_{T+1} \geq \hat{y}$ are discussed at the end of this section.

10.B.2. The Maximum Principle

- Choice variables are
  - y_t for $t = 1, 2, ...T$ and
  - z_t for $t = 0, 1, 2, ...T$.
- Let λ_t denote the multipliers for G constraints and π_{t+1} denote multipliers for Q constraints.
- In terms of economic interpretations, λ_t are usual shadow prices; π_{t+1} are shadow prices of y_{t+1}.

The Maximum Principle

Lagrangian of the full inter-temporal problem is

$$\mathcal{L} = \sum_{t=0}^{T} \left\{ F(y_t, z_t, t) + \pi_{t+1} \left[y_t + Q(y_t, z_t, t) - y_{t+1} \right] - \lambda_t G(y_t, z_t, t) \right\}$$

(Arguments in $\mathcal{L}$ are all $y_t, z_t, \lambda_t, \pi_{t+1}$)

The Maximum Principle

1. FOC with respect to z_t for $t = 0, 1, ..., T$ are

$$\partial \mathcal{L}/\partial z_t \equiv F_z(y_t, z_t, t) + \pi_{t+1} Q_z(y_t, z_t, t) - \lambda_t G_z(y_t, z_t, t) = 0.$$

2. FOC with respect to y_t for $t = 1, ..., T$ (note that y_0 and y_{T+1} are not included) are

$$\pi_{t+1} - \pi_t = -\left[F_y(y_t, z_t, t) + \pi_{t+1} Q_y(y_t, z_t, t) - \lambda_t G_y(y_t, z_t, t)\right].$$

The Maximum Principle

3. FOC with respect to π_{t+1} for $t = 0, ..., T$ are

$$\partial\mathcal{L}/\partial\pi_{t+1} = y_t + Q(y_t, z_t, t) - y_{t+1} = 0.$$

   These are Q constraints.

4. FOC with respect to λ_t for $t = 0, ..., T$ are

$$G(y_t, z_t, t) \leq 0,\ \lambda_t \geq 0 \text{ with CS.}$$

Hamiltonian

- Conditions could be written in a more compact and economic illuminating way.
- Define Hamiltonian:

$$H(y_t, z_t, \pi_{t+1}, t) = F(y_t, z_t, t) + \pi_{t+1} Q(y_t, z_t, t).$$

Hamiltonian

- Initial inter-temporal maximization problem could be rewritten as T single-period maximization problems:

$$\max_{z_t} H(y_t, z_t, \pi_{t+1}, t) \equiv \max_{z_t} F(y_t, z_t, t) + \pi_{t+1} Q(y_t, z_t, t)$$

s.t. $G(y_t, z_t, t) \leq 0$.

- In these single-period problems, only z_t are choice variables; y_t, π_{t+1} are parameters.

Hamiltonian

- Define Lagrangian:

$$L(z_t, \lambda_t, y_t \pi_{t+1}, t) = H(y_t, z_t, \pi_{t+1}, t) - \lambda_t G(y_t, z_t, t)$$

- FOCs 1 and 4 are the conditions for the single-period maximization problem.

- Write $H^*(y_t, \pi_{t+1}, t)$ resulting maximum value.

Hamiltonian

- Envelope Theorem applies to parameters y_t and π_{t+1}:

$$H_y^*(y_t, \pi_{t+1}, t) = L_y(z_t^*, \lambda_t^*, y_t, \pi_{t+1}, t)$$

$$H_\pi^*(y_t, \pi_{t+1}, t) = L_\pi(z_t^*, \lambda_t^*, y_t, \pi_{t+1}, t) = Q(y_t, z_t^*, t)$$

- Then FOCs 2 and 3 are more simply written as

$$\pi_{t+1} - \pi_t = -H_y^*(y_t, \pi_{t+1}, t)$$

$$y_{t+1} - y_t = H_\pi^*(y_t, \pi_{t+1}, t)$$

Hamiltonian

Theorem 10.1 (The Maximization Principle)**.** *First-order necessary conditions for tmaximization problem*

$$\max_{\{y_t\}_{t=1}^T,\{z_t\}_{t=0}^T} \sum_{t=0}^{T} F(y_t, z_t, t)$$

$$s.t.\ \ y_{t+1} - y_t = Q(y_t, z_t, t)$$

$$G(y_t, z_t, t) \leq 0$$

are

Theorem 9.1 (continued)

(i) for each t, z_t maximizes the Hamiltonian $H(y_t, z_t, \pi_{t+1}, t)$ subject to the single-period constraints $G(y_t, z_t, t) \leq 0$, and

(ii) the changes in y_t and π_t over time are governed by the difference equations

$$\pi_{t+1} - \pi_t = -H_y^*(y_t, \pi_{t+1}, t)$$

$$y_{t+1} - y_t = H_\pi^*(y_t, \pi_{t+1}, t)$$

Interpretations

1. For condition (i),

   - Choice of z_t affects y_{t+1} via Q constraint and therefore affects terms in objective function at $t+1$ onwards.
   - Effect of z_t on y_{t+1} equals its effect on $Q(y_t, z_t, t)$, and resulting change in objective function is shadow price π_{t+1} of y_{t+1} times $Q(y_t, z_t, t)$.
   - Hamiltonian offers a simple way to take into account future consequences of choices of controls z_t at t.

Interpretations

2. For condition (ii),

$$\pi_{t+1} - \pi_t = -H_y^*(y_t, \pi_{t+1}, t)$$

could be viewed as intertemporal no-arbitrage condition.

- Marginal unit of y_t yields
  - marginal return $F_y(y_t, z_t, t) - \lambda_t G_y(y_t, z_t, t)$ within t,
  - extra $Q_y(y_t, z_t, t)$ next period evaluated at π_{t+1}.
  - In addition, there is capital gain of $\pi_{t+1} - \pi_t$.
- When y_t is optimum, sum of these components is 0.

Transversality condition

- If we change the terminal condition to $y_{T+1} \geq 0$.
  - By Kuhn-Tucker condition for y_{T+1}, we need

$$\pi_{T+1} \geq 0 \text{ and } \pi_{T+1} y_{T+1} = 0.$$

  - That is, if any positive stocks are left, they must be worthless.

Transversality condition

- More generally, if there is a constraint $y_{T+1} \geq \hat{y}$, then we require

$$\pi_{T+1} \geq 0 \text{ and } \pi_{T+1}(y_{T+1} - \hat{y}) = 0.$$

- Such conditions on terminal stocks and respective shadow prices are called transversality conditions.

10.C. Continuous-Time Model

- We have treated time as discrete succession of periods.
- Such a formulation allows us to develop the theory using technics of Kuhn-Tucker theorem.
- However, in practice, it is sometimes more convenient to treat time as a continuous variable.

Continuous-Time Model

- We can think of continuous-time models as limit of discrete-time models when we take Δt to 0.
- Flows are now rates per time.
- In particular, Q constraint becomes

$$y(t + \Delta t) - y(t) = Q(y(t), z(t), t)\Delta t.$$

- Denote $\dot{y}(t) = \mathrm{d}y(t)/\mathrm{d}t$,

$$\dot{y}(t) = Q(y(t), z(t), t).$$

Continuous-Time Model

Problem becomes

$$\max_{y(t),z(t)} \int_0^T F(y(t), z(t), t)\mathrm{d}t$$

$$\text{s.t.} \;\; \dot{y}(t) = Q(y(t), z(t), t)$$

$$G(y(t), z(t), t) \leq 0$$

Initial condition $y(0)$ and terminal condition $y(T)$ are given.

Maximum Principle

- We also have similar results as in discrete-time model.
- Define Hamiltonian:

$$H(y(t), z(t), \pi(t), t) = F(y(t), z(t), t) + \pi(t)Q(y(t), z(t), t).$$

$\pi(t)$ is called the co-state variable.

- Lagrangian is:

$$L(z(t), \lambda(t), y(t), \pi(t), t) = H(y(t), z(t), \pi(t), t) - \lambda(t)G(y(t), z(t), t).$$

Maximum Principle

Theorem 10.2. *FOCs for the continuous-time problem is*

(i) $z(t)$ *maximizes Hamiltonian* $H(y(t), z(t), \pi(t), t)$ *subject to single period constraints* $G(y(t), z(t), t) \leq 0$*, and*

(ii) $y(t)$ *and* $\pi(t)$ *are governed by differential equations*

$$\dot{\pi}(t) = -H_y^*(y(t), \pi(t), t)$$

$$\dot{y}(t) = H_\pi^*(y(t), \pi(t), t)$$

10.D. Further Discussions

10.D.1. Infinite Horizon problems

- There is no last period in infinite horizon problems.
- So, unlike finite-horizon problems, now it is unreasonable to impose non-negative stock in last period.
- For problem to be well-defined, we need to impose transversality condition.
- Heuristically, natural extensions of previous transversality conditions for finite-horizon problems with non-negative terminal stock.

Transversality Condition

- For discrete-time problems, transversality condition is:

$$\lim_{T\to\infty} \pi_T y_T = 0.$$

- For continuous-time problems, transversality condition is:

$$\lim_{T\to\infty} \pi(T) y(T) = 0.$$

10.D.2. Present Value v.s. Current Value Hamiltonian

- In economic applications, we usually need to discount future values.
- In previous example, worker discount future utilities by β per period.
- In this section, we analyze optimization problem with discount factors explicitly expressed in objective function.
- For simplicity, we assume finite T, and y_0, y_{T+1} given.
- Transversality conditions are needed if T is infinite or we only impose $y_{T+1} \geq 0$ for finite T.

Discrete-Time Model

In order to explicitly taking into account discount factors β,

we rewrite the criterion function as:

$$\sum_{t=0}^{T} \beta^t f(y_t, z_t, t).$$

Present value Hamiltonian

- As in previous section, we define Hamiltonian, called present value Hamiltonian:

$$H^p(y_t, z_t, \pi_{t+1}, t) = \beta^t f(y_t, z_t, t) + \pi_{t+1} Q(y_t, z_t, t).$$

  π is the present value multiplier.

- Lagrangian is

$$L^p(z_t, \lambda_t, y_t, \pi_{t+1}, t) = H^p(y_t, z_t, \pi_{t+1}, t) - \lambda_t G(y_t, z_t, t)$$

Present value Hamiltonian

FOCs are:

$$L_z^p(z_t, \lambda_t, y_t, \pi_{t+1}, t) = 0$$

$$\pi_{t+1} - \pi_t = -H_y^{p^*}(y_t, \pi_{t+1}, t)$$

$$y_{t+1} - y_t = H_\pi^{p^*}(y_t, \pi_{t+1}, t) \quad \text{(Q constraint)}$$

$$G(y_t, z_t, t) \leq 0,\ \lambda_t \geq 0 \text{ with CS.}$$

Current value Hamiltonian

- It is also possible to define Hamiltonian in terms of current value:

$$H^c(y_t, z_t, \mu_{t+1}, t) = f(y_t, z_t, t) + \beta\mu_{t+1}Q(y_t, z_t, t).$$

- This is called current value Hamiltonian and μ is current value multiplier.
- Note that $H^p = \beta^t H^c$ and $\pi_t = \beta^t \mu_t$.

Current value Hamiltonian

- Lagrangian is

$$L^c(z_t, \nu_t, y_t, \mu_{t+1}, t) = H^c(y_t, z_t, \pi_{t+1}, t) - \nu_t G(y_t, z_t, t).$$

- Note that $L^p = \beta^t L^c$ and $\lambda_t = \beta^t \nu_t$.

Current value Hamiltonian

From FOCs for present value Hamiltonian, we could deduce FOCs for current value Hamiltonian:

$$L_z^c(z_t, \nu_t, y_t, \mu_{t+1}, t) = 0$$

$$\beta \mu_{t+1} - \mu_t = -H_y^{c^*}(y_t, \mu_{t+1}, t)$$

$$\beta(y_{t+1} - y_t) = H_\mu^{c^*}(y_t, \mu_{t+1}, t) \quad \text{(Q constraint)}$$

$$G(y_t, z_t, t) \leq 0, \ \nu_t \geq 0 \text{ with CS.}$$

Continuous-Time Model

- Discount factor changes from $\beta \equiv \frac{1}{1+\rho}$ to $e^{-\rho t}$ where ρ is discount rate:

$$\lim_{n\to\infty}\left(\frac{1}{1+\frac{\rho}{n}}\right)^{n} = \lim_{n\to\infty}\left[\left(1+\frac{\rho}{n}\right)^{\frac{n}{\rho}}\right]^{-\rho} = e^{-\rho}.$$

- Thus, we rewrite criterion function as:

$$\int_0^T e^{-\rho t} f(y(t), z(t), t)\mathrm{d}t.$$

Present value Hamiltonian

- We define present value Hamiltonian:

$$H^p(y(t), z(t), \pi(t), t) = e^{-\rho t} f(y(t), z(t), t) + \pi(t) Q(y(t), z(t), t).$$

π is present value multiplier.

- Lagrangian is

$$L^p(z(t), \lambda(t), y(t), \pi(t), t) = H^p(y(t), z(t), \pi(t), t) - \lambda(t) G(y(t), z(t), t).$$

Present value Hamiltonian

FOCs are:

$$L_z^p(z(t), \lambda(t), y(t), \pi(t), t) = 0$$

$$\dot{\pi}(t) = -H_y^{p^*}(y(t), \pi(t), t)$$

$$\dot{y}(t) = H_\pi^{p^*}(y(t), \pi(t), t)$$

$$G(y(t), z(t), t) \leq 0,\ \lambda(t) \geq 0 \text{ with CS.}$$

Current value Hamiltonian

- It is also possible to define Hamiltonian in terms of current value:

$$H^c(y(t), z(t), \pi(t), t) = f(y(t), z(t), t) + \mu(t) Q(y(t), z(t), t).$$

- This is called current value Hamiltonian and μ is current value multiplier.
- Note that $H^p = e^{-\rho t} H^c$ and $\pi(t) = e^{-\rho t} \mu(t)$.

Current value Hamiltonian

- Lagrangian is

$$L^c(z(t),\nu(t),y(t),\pi(t),t) = H^c(y(t),z(t),\pi(t),t) - \nu(t)G(y(t),z(t),t).$$

- Note that $L^p = e^{-\rho t}L^c$ and $\lambda(t) = e^{-\rho t}\nu(t)$.

Current value Hamiltonian

From the FOCs for present value Hamiltonian, we could deduce FOCs for current value Hamiltonian:

$$L_z^c(z(t), \nu(t), y(t), \pi(t), t) = 0$$

$$\dot{\mu}(t) - \rho\mu(t) = -H_y^{c^*}(y(t), \mu(t), t)$$

$$\dot{y}(t) = H_\mu^{c^*}(y(t), \mu(t), t)$$

$$G(y(t), z(t), t) \leq 0,\ \lambda(t) \geq 0 \text{ with CS.}$$

10.E. Examples

Example 1: Life-Cycle Saving

- Consider continuous-time version of life-cycle saving model.
- Evolution of k is governed by

$$\dot{k} = w + rk - c.$$

- w is constant wage rate and r is constant interest rate.
- Assume no inheritances or bequests: $k(0) = k(T) = 0$.

Example 1: Life-Cycle Saving

- Instantaneous utility function is $\ln(c)$
- Discount rate is ρ
- Objective function is:

$$\int_0^T e^{-\rho t} \ln(c) \mathrm{d}t.$$

Maximum Principle

- Define Hamiltonian:

$$H = e^{-\rho t} \ln(c) + \pi(w + rk - c).$$

- Condition for c is:

$$H_c = e^{-\rho t} c^{-1} - \pi = 0 \implies c^* = e^{-\rho t} \pi^{-1}$$

- Substituting into H:

$$H^* = e^{-\rho t}[-\rho t - \ln(\pi)] + \pi(w + rk) - e^{-\rho t}.$$

Maximum Principle

- Differential equations for k and π are:

$$\dot{\pi} = -H_k^* = \pi r$$

$$\dot{k} = H_\pi^* = w + rk - e^{-\rho t}\pi^{-1}$$

Example 2: Optimum Growth

- Consider optimal saving problem from the view of economy as a whole.
  - Rate of return now is endogenous.
  - Besides, consider $T = \infty$.
- k denotes stock of capital.
- Let $F(k)$ be production function.
  - F is increasing, strictly concave with $F(0) = 0$ and $F'(0) = \infty$.
- Capital depreciates at a constant rate δ.

Optimum Growth

- c is the consumption flow.
- Capital accumulation equation is

$$\dot{k} = F(k) - \delta k - c. \tag{10.1}$$

- Initial capital stock $k(0)$ is given.

Optimum Growth

- Objective is still to maximize present discounted value of utilities:

$$\int_0^\infty e^{-\rho t} U(c) \mathrm{d}t,$$

  - Flow utility $U(c)$ is increasing and strictly concave.

Maximum Principle

- Define Hamiltonian:

$$H = e^{-\rho t}U(c) + \pi(F(k) - \delta k - c).$$

- Condition for c is:

$$H_c = e^{-\rho t}U'(c) - \pi = 0 \implies e^{-\rho t}U'(c) = \pi.$$

- Condition for k is:

$$\dot{\pi} = -H_k^* = -\pi(F'(k) - \delta)$$

Maximum Principle

- Condition for π gives capital accumulation equation (10.1) at optimal c.
- Furthermore, this is an infinite-horizon problem, so we require transversality condition:

$$\lim_{T\to\infty} \underbrace{\pi(T)}_{\text{discounted shadow price}} k(T) = 0 \implies \lim_{T\to\infty} e^{-\rho T} U'(c(T))k(T) = 0.$$

  - It means that present value of capital stock in infinite future is zero.

Analysis

We could derive a pair of differential equations in k and c:

$$\dot{k} = F(k) - \delta k - c$$

and Euler equation

$$\frac{\dot{c}}{c} = \frac{F'(k) - (\rho + \delta)}{\eta(c)}$$

where $\eta(c) = -\frac{cU''(c)}{U'(c)}$ is the elasticity of marginal utility of consumption.

Phase Diagram