Chapter 9. Uncertainty

This chapter concerns choice under uncertainty. It is an important topic in economics, and is of great practical interest. In real-life, almost every decision needs to be made under uncertainty. For example, when you make up your decision to learn the current course, you will only have some estimates about its usefulness. In the end, it may be more or less useful than you thought.

In this chapter, we will sketch a systematic way of making such decisions. In terms of mathematics, there will be nothing new. We will only use the tactics we learned in the previous chapters. However, you will see more economic concepts and intuitions.

Let's get familiarized with the problems of choice under uncertainty. As illustrated in the course-choosing example, uncertainty means that you do not anticipate a sure outcome. To make our discussion more concrete, we will need to introduce some concepts. First, consider the following simple example:

Example 9.1. Suppose that you have access to the following lottery: the lottery pays \$100 with probability 1/4, and pays nothing with the remaining probability. The question is, do you wish to pay \$25 for such a lottery?

This is a problem of choice under uncertainty, because the outcome is uncertain: in the end, you will either get \$100 or \$0. There are two important elements in the problem:

(i) *The outcomes.* In the example, the outcomes refer to the state *paying \$100*, and the state *paying \$0*.

(ii) *The probabilities associated with the outcomes.* In the example, 1/4 is the probability associated with the state *paying \$100*, and 3/4 is the probability associated with the state *paying \$0*.

Note that the probabilities are *objective* here, but they could be referred to as *subjective probabilities* in certain applications. Besides, the probabilities in a well-defined problem should be non-negative, and add up to 1. In this example, we could write the consumer's utility from the lottery could be written as follows: $U(\$100, \$0; 1/4, 3/4)$.

More generally, denote the possible outcomes by $Y_1, Y_2, ..., Y_m$, and the probability associated with the outcomes by $p_1, p_2, .., p_m$. The utility could be written as

$$U(Y_1, Y_2, ..., Y_m; p_1, p_2, .., p_m).$$

Next, we will introduce a widely-used method to express the utility in a way that more analysis could be performed.

9.A. Expected Utility

Since probabilities are involved, it is somewhat natural to make use of mathematical expectation, or probability weighted average. For instance, in Example 9.1, we could express the utility as follows:

$$U(\$100, \$0; 1/4, 3/4) = 1/4U(\$100) + 3/4U(\$0).$$

This is called the *von Neumann-Morgenstern utility function*, and is of *expected utility* form. For a general utility function, the expected utility form is expressed as follows:

$$U(Y_1, Y_2, ..., Y_m; p_1, p_2, .., p_m) = p_1 U(Y_1) + p_2 U(Y_2) + ... + p_m U(Y_m) = \sum_{i=1}^{m} p_i U(Y_i). \quad (9.1)$$

This formulation is very useful in its simplicity and its ability to capture economically interesting aspects of behavior. We will discuss some implications of this representation.[1]

Risk-aversion. Now consider Y_i's as money amounts. Since more money is better, U is an increasing function. The definition of *Risk-aversion* is intuitive. In Example 9.1, the lottery gives in expectation $\$100 \times 1/4 + \$0 \times 3/4 = \$25$. A risk-averse individual dislikes risk, and thus prefers the sure outcome of \$25 to the lottery that gives on average \$25. In general, for two distinct outcomes Y_1 and Y_2 with (any) positive probability p and $(1-p)$ respectively, a decision maker is risk-averse if

$$U(pY_1 + (1-p)Y_2) > pU(Y_1) + (1-p)U(Y_2).$$

This is, U is (strictly) concave.

[1] The applicability of expected utility is out of scope of this course, and thus will not be discussed.

More generally, we could include more than 2 states: A decision maker is risk-averse if

$$U(\sum_{i=1}^{m} p_i Y_i) > \sum_{i=1}^{m} p_i U(Y_i). \tag{9.2}$$

If U is twice differentiable, $U'' < 0$ corresponds to risk-aversion.

Insurance. Let's now bring back the decision variable x, which affect some or all of the outcomes and probabilities. Suppose $Y_1 < Y_2$, which means that the first state entails some loss relative to the second. Y_1 occurs with probability p and Y_2 with probability $(1-p)$. A risk-averse decision maker would want to purchase insurance. Consider an insurance policy that requires an advance payment of x (paid independent of the state realization), and gives X if state 1 is realized. Suppose that the insurance policy is actuarially fair: $pX = x$.[2] The decision maker's objective function is

$$\begin{aligned} &\max_{x\geq 0} pU(Y_1 - x + X) + (1-p)U(Y_2 - x) \\ \iff &\max_{x\geq 0} pU(Y_1 - x + x/p) + (1-p)U(Y_2 - x) \end{aligned}$$

The first-order condition for x gives:

$$\begin{aligned} &pU'(Y_1 - x + x/p)(1/p - 1) - (1-p)U'(Y_2 - x) \leq 0 \text{ and } x \geq 0, \\ &\text{with complementary slackness.} \end{aligned} \tag{9.3}$$

When $x = 0$, $pU'(Y_1)(1/p-1)-(1-p)U'(Y_2) = (1-p)[U'(Y_1)-U'(Y_2)] \underbrace{>}_{U''<0} 0$, contradicting with (9.3). Therefore, we must have $x > 0$ at the optimum. By (9.3),

$$\begin{aligned} &pU'(Y_1 - x + x/p)(1/p - 1) - (1-p)U'(Y_2 - x) = 0 \\ \Longrightarrow\ &U'(Y_1 - x + x/p) = U'(Y_2 - x) \end{aligned} \tag{9.4}$$

When $U'' < 0$, the objective function is concave in x and the first-order condition is also sufficient. The first-order condition (9.4) implies $Y_1 - x + x/p = Y_2 - x$. This is the **full-insurance** result: a risk-averse decision maker would buy the actuarially fair insurance to the point where the outcomes in different states are equal.

[2] It is an outcome of a perfectly competitive insurance industry. Insurance company could pool a large number of independent risks, and is thus considered to be risk-free. Zero-profit condition in such an industry is exactly $pX = x$.

Care. Consider again the previous problem faced by the decision maker, but leave aside insurance for the moment. Now suppose that the probability of the bad outcome (state 1) can be reduced by incurring an expense z in advance. Specifically, you could think of it as exercising more care by yourself to reduce the probability of being ill. In terms of modelling, we make the probability p a function of z, and the function is decreasing. Since p is bounded below, it will generally be convex. The objective function is

$$\max_{z \geq 0} \phi(z) \equiv \max_{z} p(z)U(Y_1 - z) + (1 - p(z))U(Y_2 - z)$$

Then, derivative of $\phi(z)$ gives

$$\begin{aligned}\phi'(z) = & \underbrace{\underbrace{-p'(z)}_{\text{reduction of prob.}} \underbrace{[U(Y_2 - z) - U(Y_1 - z)]}_{\text{utility diff.}}}_{\text{marginal benefit}} \\ & - \underbrace{\{p(z)U'(Y_1 - z) + (1 - p(z))U'(Y_2 - z)\}}_{\text{marginal cost}}\end{aligned}$$

The optimal solution is defined by the first-order condition[3]:

$$\phi'(z^*) = 0.$$

Moral Hazard. Now suppose both insurance and care variables are available. The interaction between the insurance company and the decision maker could be formulated as the game below:

1. Insurance company sells the insurance at constant rate $p(\bar{z})$ per \$1 coverage. That is, if the individual purchases x shares of insurance, the insurance company would pay the individual $X = \frac{x}{p(\bar{z})}$ when the bad outcome (state 1) occurs.
2. The decision maker chooses how much to purchase x.
3. The decision maker chooses the care parameter z.
4. Outcome realized and the decision maker gets paid from the insurance company if the realized state is 1.

We assume that the insurance policy is actuarially fair (in equilibrium).

[3] We impose the assumption $\phi'(0) \to -\infty$ to ensure an interior solution. Similar assumptions are usually made in economic applications.

Since the insurance policy is actuarially fair, we have $p(z^*)X = x$, where z^* is decision maker's actual choice. Plugging in $X = \frac{x}{p(\bar{z})}$ gives $\bar{z} = z^*$.

The objective function for the decision maker is

$$\max_{x\geq 0, z\geq 0} \phi(x,z) \equiv \max_{x\geq 0, z\geq 0} p(z)U(Y_1 - z - x + x/p(z^*)) + (1-p(z))U(Y_2 - z - x)$$

Partial derivative of $\phi(x,z)$ with respect to x gives

$$\phi_x(x,z) = p(z)U'(Y_1 - z - x + x/p(z^*))(1/p(z^*) - 1) - (1-p(z))U'(Y_2 - z - x)$$

Next, we show that the optimal $x^* > 0$ must hold.

$$\begin{aligned}\phi_x(0,z^*) &= p(z^*)U'(Y_1 - z^*)(1/p(z^*) - 1) - (1-p(z^*))U'(Y_2 - z^*)\\ &= (1-p(z^*))[U'(Y_1 - z^*) - U'(Y_2 - z^*)] \underbrace{>}_{U''<0} 0.\end{aligned}$$

Since $\phi_x(0,z^*) > 0$, marginally increase x will increase $\phi(x,z^*)$, and therefore, $x^* > 0$ must hold. First-order condition on x gives

$$\begin{aligned}&\phi_x(x^*,z^*) = 0\\ \Longrightarrow\ &U'(Y_1 - z^* - x^* + x^*/p(z^*)) = U'(Y_2 - z^* - x^*)\\ \Longrightarrow\ &Y_1 - z^* - x^* + x^*/p(z^*) = Y_2 - z^* - x^* \qquad (9.5)\end{aligned}$$

Therefore, the optimal choices of x^* and z^* must satisfy (9.5) above. Let $Y_1 - z^* - x^* + x^*/p(z^*) = Y_2 - z^* - x^* = Y_0$. Partial derivatvie of $\phi_z(x,z)$ with respect to z gives

$$\begin{aligned}\phi_z(x,z) = &\underbrace{-p'(z)\left[U(Y_2 - z - x) - U(Y_1 - z - x + x/p(z^*))\right]}_{\text{marginal benefit}}\\ &- \underbrace{[p(z)U'(Y_1 - z - x + x/p(z^*)) + (1-p(z))U'(Y_2 - z - x)]}_{\text{marginal cost}}\end{aligned}$$

Evaluated at the optimal level (x^*, z^*), we have

$$\begin{aligned}\phi_z(x^*,z^*) = &-p'(z^*)\underbrace{\left[U(Y_2 - z^* - x^*) - U(Y_1 - z^* - x^* + x^*/p(z^*))\right]}_{=0}\\ &- \underbrace{[p(z^*)U'(Y_1 - z^* - x^* + x^*/p(z^*)) + (1-p(z^*))U'(Y_2 - z^* - x^*)]}_{=U'(Y_0)}\\ = &-U'(Y_0) < 0.\end{aligned}$$

Since $\phi_z(x^*,z^*) < 0$, marginally decrease z^* will increase $\phi(x^*,z^*)$, and therefore, the optimum of care occurs at the corner $z^* = 0$. This is known as *"moral hazard"*: the availability of full insurance destroys the incentive to exercise costly care.

In the subsections that follow, we will study portfolio choice. Since we will be working with multiple states, for convenience, we introduce a continuous representation. The index i is replaced by a continuous random variable r with support $[\underline{r}, \overline{r}]$. The expected utility form in (9.1) is modified by replacing probabilities with densities, and sums with integrals:

$$\mathbb{E}[U(Y)] = \int_{\underline{r}}^{\overline{r}} U(Y(r))f(r)\mathrm{d}r.$$

The interpretation of risk-aversion parallels (9.2): A decision-maker is risk averse if

$$U(\mathbb{E}(Y)) > \mathbb{E}[U(Y)] \iff U(\int_{\underline{r}}^{\overline{r}} Y(r)f(r)\mathrm{d}r) > \int_{\underline{r}}^{\overline{r}} U(Y(r))f(r)\mathrm{d}r.$$

9.B. One Safe and One Risky Asset

A *risk-averse* investor has initial wealth W_0, and has the following two investment options:

(i) A risky asset: investing x gives $x(1+r)$, where r is a random variable with density $f(r)$ and support $[\underline{r}, \overline{r}]$; Assume $\mathbb{E}[r] > 0$ and $\underline{r} < 0$, so that the risky asset does not always generate a positive return, but on average the return is positive.

(ii) A safe asset: investing x gives x.

Therefore, investing $x \in [0, W_0]$ in the risky asset and the rest in the safe asset generates final wealth

$$W = x(1+r) + (W_0 - x) = W_0 + xr.$$

The investor's objective is to maximize the expected final wealth:

$$\max_{x\in[0.W_0]} \mathbb{E}[(U(W))] \equiv \max_{x\in[0.W_0]} \int_{\underline{r}}^{\overline{r}} U(W_0 + xr)f(r)\mathrm{d}r.$$

Let $\phi(x) = \mathbb{E}[(U(W))]$. Derivative of $\phi(x)$ gives

$$\phi'(x) = \int_{\underline{r}}^{\overline{r}} rU'(W_0 + xr)f(r)\mathrm{d}r.$$

Note when $x = 0$:

$$\phi'(0) = \int_{\underline{r}}^{\overline{r}} rU'(W_0)f(r)\mathrm{d}r = U'(W_0)\int_{\underline{r}}^{\overline{r}} rf(r)\mathrm{d}r = U'(W_0)\mathbb{E}[r] > 0.$$

So, $x = 0$ is not optimal. Therefore, the risk-averse investor will buy at least some of the actuarially good investment.

Typically, the investor will hold some of each asset. The first-order condition is

$$\phi'(x) = \int_{\underline{r}}^{\overline{r}} rU'(W_0 + xr)f(r)\mathrm{d}r = 0. \tag{9.6}$$

If there is an $x < W_0$ satisfying this, then strict concavity of U guarantees that it is the global maximum:

$$\phi''(x) = \int_{\underline{r}}^{\overline{r}} r^2 \underbrace{U''(W_0 + xr)}_{U''(W)<0 \text{ for all } W} f(r)\mathrm{d}r < 0. \tag{9.7}$$

Next, assuming that there exists an interior maximum, we consider the comparative statics of x with respect to W_0. That is, whether the investor would invest more or less in the risky asset when he becomes wealthier. Now, we recognize W_0 as a parameter in ϕ, i.e., $\phi(x, W_0)$. Then, first-order condition for an interior solution is $\phi_x(x, W_0) = 0$. Total differentiation gives

$$\phi_{xx}(x, W_0)\mathrm{d}x + \phi_{xw}(x, W_0)\mathrm{d}W_0 = 0 \implies \mathrm{d}x/\mathrm{d}W_0 = -\phi_{xw}(x, W_0)/\phi_{xx}(x, W_0).$$

By second-order sufficient condition, $\phi_{xx}(x, W_0) < 0$.

Then, the sign of $\mathrm{d}x/\mathrm{d}W_0$ is the same as the numerator on the right-hand side:

$$\phi_{xw}(x, W_0) = \int_{\underline{r}}^{\overline{r}} rU''(W_0 + xr)f(r)\mathrm{d}r.$$

Since $\underline{r} < 0 < \overline{r}$, we could not decide the sign of $\phi_{xw}(x, W_0)$ in general.

To gain more insight, we introduce a measure of risk-aversion, called *absolute risk-aversion*, and denoted by $A(W)$:

$$A(W) = -U''(W)/U'(W). \tag{9.8}$$

Experimental and empirical evidence is consistent with $A(W)$ being decreasing in W.[4] If this is the case, then we would be able to show $\phi_{xw}(x, W_0) > 0$.

[4] Friend, I., & Blume, M. E. (1975). The Demand for Risky Assets. *American Economic Review*, 65(5), 900-922.

Here comes the detailed reasoning:

(i) For $r < 0$,

$$\begin{aligned}
&-U''(W_0+xr)/U'(W_0+xr) > -U''(W_0)/U'(W_0) = A(W_0)\\
\Longrightarrow\ & rU''(W_0+xr)/U'(W_0+xr) > -rA(W_0)\\
\Longrightarrow\ & rU''(W_0+xr) > -rA(W_0)U'(W_0+xr).
\end{aligned}$$

(ii) For $r > 0$,

$$\begin{aligned}
&-U''(W_0+xr)/U'(W_0+xr) < -U''(W_0)/U'(W_0) = A(W_0)\\
\Longrightarrow\ & -rU''(W_0+xr)/U'(W_0+xr) < rA(W_0)\\
\Longrightarrow\ & rU''(W_0+xr) > -rA(W_0)U'(W_0+xr).
\end{aligned}$$

Then, $rU''(W_0+xr) > -rA(W_0)U'(W_0+xr)$ for all $r \neq 0$.

Therefore,

$$\begin{aligned}
\phi_{xw}(x, W_0) &= \int_{\underline{r}}^{\overline{r}} rU''(W_0+xr)f(r)\mathrm{d}r > \int_{\underline{r}}^{\overline{r}} -rA(W_0)U'(W_0+xr)f(r)\mathrm{d}r\\
&= -A(W_0)\underbrace{\int_{\underline{r}}^{\overline{r}} rU'(W_0+xr)f(r)\mathrm{d}r}_{=0 \text{ by Equation (9.6)}} = 0.
\end{aligned}$$

Thus, $\phi_{xw}(x, W_0) > 0$, which implies $\mathrm{d}x/\mathrm{d}W_0 > 0$. That is, the investor would invest more in the risky asset when he becomes wealthier. Remember that we need the absolute risk-aversion $A(W)$ being a decreasing function for this result to hold.

9.C. Examples

Example 9.1: Managerial Incentives. A risk-neutral owner (she) has to hire a risk-neutral manager (he) to run a project. If the project succeeds, it will produce value V. If the project fails, it generates no return. The probability of success depends on manager's effort. Given that the manager exerts effort, the project will succeed with probability p. No effort will reduce the probability to $q < p$. Effort cost is e To make it worthwhile to exert effort, suppose that exerting effort generates higher total surplus:

$$pV - e > qV \implies (p - q)V > e. \tag{9.9}$$

Assume that the manager's outside job pays him w.

What is optimal compensation scheme when

(i) The owner can observe manager's effort?

(ii) The owner cannot observe manager's effort?

Solution.

Case I: Observable Effort.

In this case, the owner could compensate effort directly. Since inducing effort generates higher total surplus, the owner would be willing to do so as long as it is not too expensive to attract the manager. Let the payment to manager be W, paid when the manager exerts effort. The manager is willing to work for the owner and exert effort if

$$W - e \geq w \implies W \geq e + w.$$

To get more profit, the owner would pay the least amount $W = e + w$. After paying W to the manager, the owner gets $pV - W = pV - e - w$. The owner thus is willing to hire the manager if

$$w < pV - e. \tag{9.10}$$

(9.10) is an assumption that we would make throughout the analysis, since otherwise, the manager would not be hired.

Under the assumption, the owner could offer $w + e$ to the manager, and demand effort in return. The owner would get $pV - e - w$ and the manager gets $w + e - e = w$, the same as what he would get from the outside job.

Case II: Unobservable Effort.

In this case, compensating effort directly would not work. To see this, suppose that compensation is still based on (now unoberservable) effort, then the manager could lie about effort: the manager could promise to exert effort, but shirk instead. Because of unobservability of effort and the probabilistic nature of the outcome, the owner would not catch such a lie.

Therefore, the best thing the owner could do is to base his payment scheme on the thing that he could observe, i.e., the outcome. Suppose that the owner pays the manager x if the project succeeds, and y if it fails. Two constraints need to be satisfied:

(i) Given such a payment scheme, the manager would exert effort if

$$\begin{aligned} & px + (1-p)y - e \geq qx + (1-q)y \\ \Longrightarrow\ & (p-q)(x-y) \geq e. \end{aligned} \tag{IC}$$

This is called the *incentive compatibility constraint.*

(ii) The manager will agree to work if

$$\begin{aligned} & px + (1-p)y - e \geq w \\ \Longrightarrow\ & y + p(x-y) \geq w + e. \end{aligned} \tag{IR}$$

This is called the *participation constraint*, or *individual rationality constraint.*

Thus, the owner's problem is to maximize her profit subject to constraints (IC) and (IR).

$$\begin{aligned} & \max_{x,y} pV - [px + (1-p)y] \equiv \max_{x,y} pV - y - p(x-y) \\ & \text{s.t. } (IC)\ \&\ (IR) \end{aligned}$$

(IR) must be binding since otherwise, we could decrease x and y by the same sufficiently small amount so that (IR) is still satisfied. After the change, (IC) is unaffected, however the objective function gets larger. So, the payment scheme with non-binding (IR) must not be optimal, and thus (IR) must be binding.

From IC and binding IR, we get

$$y^* \leq w - \frac{eq}{p-q} \text{ and } x^* \geq w + \frac{e(1-q)}{(p-q)}.$$

Therefore, one interpretation is that the manager's compensation consists of the basic salary w, plus a reward for success and minus a penalty for failure. By binding IR, the owner's expected profit is

$$\pi = pV - y^* - p(x^* - y^*) = pV - w - e,$$

the same as when she could observe the manager's effort directly.

However, one potential problem here is that $y^*_{\max} = w - eq/(p-q)$ is not guaranteed to be positive. $y^* < 0$ means that the payment scheme would involve a fine under failure, which is always not feasible.

Suppose $y^*_{\max} = w - eq/(p-q) < 0$ and fine is not allowed, i.e., $y \geq 0$ is required. The solution is to go as far as possible, i.e., $y = 0$. (IC) and (IR) becomes

$$(p-q)(x-0) \geq e \implies x \geq \frac{e}{p-q}; \tag{IC'}$$

$$0 + p(x-0) \geq w + e \implies x \geq \frac{w+e}{p}. \tag{IR'}$$

The problem becomes:

$$\max_x pV - 0 - p(x-0) \equiv \max_x pV - px$$
$$\text{s.t. } (IC') \text{ \& } (IR')$$

The owner wants x to be as small as possible.

From $y^*_{\max} = w - eq/(p-q) < 0$, we have

$$(w+e)/p < e/(p-q). \tag{9.11}$$

Therefore, the minimum x is

$$x^{**} = e/(p-q).$$

And the profit becomes

$$\pi = pV - px^{**} = pV - pe/(p-q).$$

By (9.11), this profit level is lower compared to the previous cases, $\pi = pV - w - e$.

By (9.9), this profit level is still positive.

Example 9.2: Cost-Plus Contracts. This example is motivated by the cost-plus contract. Government expenditures are often made on such a cost-plus basis, that is, the government reimburses the supplier's cost plus a normal profit. In this example, we are concerned with the appropriated amount of reimbursement when the government does not observe the supplier's cost.

Suppose the true average cost of production can take just two values: c_1 and c_2 (normal profit included), with $0 < c_1 < c_2$. We call the supplier with cost c_i Type-i supplier.

The supplier is privately informed of its own type. Before contracting, the government's estimate of the probability of the supplier being Type-1 is β_1, and Type-2 is $\beta_2 = 1 - \beta_1$. The problem here is that the low cost supplier would pretend to be of high cost and get more reimbursement from the government. To mitigate the problem, the government could purchase different amounts and offer distinct payments, depending on the cost declared by the supplier. More specifically, the government would offer the following contracts: if the supplier claims to have cost c_i for $i = 1, 2$, the government purchases q_i units and pays R_i. In game theory, the use of different contracts to separate supplier types is called "screening".[5]

The governments gets benefit $B(q)$ from quantity q. $B(q)$ is strictly increasing, strictly concave in q, and

$$B'(0) > c_2 \tag{A}$$

so that the government would demand positive quantities from a supplier of either type if cost can be observed.

What is the optimal menu of contracts (q_1, R_1) and (q_2, R_2) when cost is unobservable?

Solution. Before analyzing the problem with unobservable cost, we will first analyze the problem with observable cost.

Observable Cost. When cost is observable, the government could design contract based on the supplier's true type. The government's problem facing a supplier with Type-i is:

$$\max_{q_i, R_i} B(q_i) - R_i$$

$$\text{s.t.} \ \ R_i - c_i q_i \geq 0. \tag{IR}$$

$$q_i \geq 0, R_i \geq 0.$$

The government's objective is to maximize the benefit from purchase of q_i minus the reimbursement R_i. The constraint (IR) is the supplier's *participation constraint*, or *individual rationality* constraint. This constraint ensures that the supplier is better-off accepting

[5] More generally, "screening models" refers to the situation where an agent (the supplier here) has private information about his type before the principal (the government here) makes a contract offer. The principal will then offer a menu of contracts in order to separate the different types.

the government's contract, which pays R_i and incurs the cost of $c_i q_i$, compared to not contracting with the government, which generates 0.

Before solving the problem, we make two observations:

1. (IR) must be binding. Otherwise, the government could demand a higher R_i without violating the constraints, and the objective function becomes larger.
2. (IR) and $q_i \geq 0$ implies $R_i \geq 0$.

The government's problem is reduced to

$$\max_{q_i} B(q_i) - c_i q_i$$
$$\text{s.t. } q_i \geq 0.$$

Next, we solve the problem using the Lagrange's theorem.

- Form the Lagrangian:
$$\mathcal{L}(q_i) = B(q_i) - c_i q_i.$$
- The first-order necessary condition is
$$\partial \mathcal{L} / \partial q_i = B'(q_i) - c_i \leq 0 \text{ and } q_i \geq 0 \text{ with complementary slackness.}$$

By Assumption (A) and the strict concavity of $B(q)$, we have $q_i > 0$ and

$$B'(q_i) = c_i. \tag{9.12}$$

Thus, the optimal q_i is given by (9.12). After obtaining q_i, the optimal R_i is given by binding (IR).

Unobservable Cost. As indicated in the question, when cost is unobservable, the government would offer two contracts and let the supplier choose.

To make the supplier willing to choose the contract designed for his type, we must ensure that Type-1 supplier prefers its contract to Type-2's, and similarly, Type-2 supplier prefers its contract to Type-1's:

$$R_1 - c_1 q_1 \geq R_2 - c_1 q_2; \tag{IC_1}$$
$$R_2 - c_2 q_2 \geq R_1 - c_2 q_1. \tag{IC_2}$$

These are the *incentive compatibility constraints*.

Moreover, we need to ensure that the supplier would want to participate:

$$R_1 - c_1 q_1 \geq 0; \tag{IR_1}$$

$$R_2 - c_2 q_2 \geq 0. \tag{IR_2}$$

These are the *participation constraints*.

We also need to ensure $q_i \geq 0$ and $R_i \geq 0$. The government's problem is

$$\begin{aligned} \max_{q_1,q_2,R_1,R_2} \beta_1 \left[B(q_1) - R_1\right] + \beta_2 \left[B(q_2) - R_2\right] \\ \text{s.t. } (IC_1),\ (IC_2),\ (IR_1),\ (IR_2) \\ q_1 \geq 0, q_2 \geq 0, R_1 \geq 0, R_2 \geq 0. \end{aligned}$$

It is a maximization problem with 4 inequality constraints and 4 non-zero variables. These inequality pairs permit $2^8 = 256$ patterns of equations. Solving the problem directly involves a lot of work. So, we will make some initial analysis to simplify the problem.

Lemma 1. *$R_i \geq 0$ is implied by (IR_1), (IR_2) and $q_i \geq 0$.*

Proof. We take R_1 as an example.

$$R_1 \underbrace{\geq}_{(IR_1)} c_1 q_1 \underbrace{\geq}_{q_1 \geq 0} 0.$$

$R_2 \geq 0$ follows similarly. □

So, we could safely ignore the non-negativity constraints on R_i.

Lemma 2. *(IR_1) is implied by (IC_1), (IR_2) and $q_2 \geq 0$.*

Proof. We want to show that (IR_1) holds, i.e., $R_1 - c_1 q_1 \geq 0$.

$$R_1 - c_1 q_1 \underbrace{\geq}_{(IC_1)} R_2 - c_1 q_2 \underbrace{\geq}_{(IR_2)} c_2 q_2 - c_1 q_2 = (c_2 - c_1) q_2 \underbrace{\geq}_{q_2 \geq 0,\ c_1 < c_2} 0$$

(IR_1) is implied. □

So, we could safely ignore (IR_1).

From Lemma 1 and Lemma 2, the government's problem could be simplified as follows:

$$\max_{q_1,q_2,R_1,R_2} \beta_1 [B(q_1) - R_1] + \beta_2 [B(q_2) - R_2]$$

$$\text{s.t. } R_1 - c_1 q_1 \geq R_2 - c_1 q_2; \tag{IC_1}$$

$$R_2 - c_2 q_2 \geq R_1 - c_2 q_1; \tag{IC_2}$$

$$R_2 - c_2 q_2 \geq 0; \tag{IR_2}$$

$$q_1 \geq 0, q_2 \geq 0.$$

Lemma 3. *(IR_2) must be binding in the optimal scheme, i.e., $R_2 - c_2 q_2 = 0$.*

Proof. Suppose not, i.e., $R_2 - c_2 q_2 > 0$, then q_1 and q_2 can be slightly raised by the same amount ε so that (IR_2) still holds. After such a change,

(i) All constraints still holds. Such change will affect the left-hand side and the right-hand side of (IC_1) and (IC_2) equally, so the inequalities still hold. The non-negativity constraints on q_i still hold as q_1 and q_2 become larger.

(ii) The objective function gets larger since $B(q)$ is strictly increasing.

Therefore, a scheme with $R_2 - c_2 q_2 > 0$ must not be optimal. In other words, (IR_2) must be binding in the optimal scheme. □

Lemma 4. *(IC_1) must be binding in the optimal scheme, i.e., $R_1 - c_1 q_1 = R_2 - c_1 q_2$.*

Proof. Suppose not, i.e., $R_1 - c_1 q_1 > R_2 - c_1 q_2$. Then q_1 can be slightly raised by ε so that (IC_1) still holds. After such a change,

(i) All constraints still hold. Such a change will not affect (IR_2) and $q_2 \geq 0$. The non-negativity constraints on q_1 still hold as q_1 gets larger. For (IC_2), the right-hand side gets smaller and the inequality still holds.

(ii) The objective function gets larger since $B(q_1)$ is strictly increasing.

Therefore, a scheme with $R_1 - c_1 q_1 > R_2 - c_1 q_2$ must not be optimal. In other words, (IC_1) must be binding in the optimal scheme. □

By Lemma 3 and Lemma 4, we have

$$R_2 = c_2 q_2 \tag{R_2}$$

$$R_1 = c_1 q_1 + (c_2 - c_1) q_2 \tag{R_1}$$

Plugging (R_2) and (R_1) into (IC_2), we have

$$R_2 - c_2 q_2 \geq R_1 - c_2 q_1 \underbrace{\iff}_{(R_2),\ (R_1)} (c_2 - c_1)(q_1 - q_2) \geq 0 \underbrace{\iff}_{c_1 < c_2} q_1 \geq q_2 \qquad (IC_2\text{'})$$

Plugging (R_2) and (R_1) into the objective function, and recognizing the remaining constraints, the maximization problem is simplifed as follows:

$$\max_{q_1, q_2} \beta_1 \left[B(q_1) - (c_1 q_1 + (c_2 - c_1) q_2)\right] + \beta_2 \left[B(q_2) - c_2 q_2\right]$$

$$\text{s.t.}\ \ q_1 \geq q_2 \qquad (IC_2\text{'})$$

$$q_2 \geq 0$$

We could solve the maximization problem in the usual way. See Appendix A.

Here, we introduce another way of solving the problem. We first solve the relaxed problem with no (IC_2'), and then show that the solution to the relaxed problem satisfies (IC_2') and is thus also the solution to the initial problem.

The relaxed problem is as follows:

$$\max_{q_1, q_2} \beta_1 \left[B(q_1) - (c_1 q_1 + (c_2 - c_1) q_2)\right] + \beta_2 \left[B(q_2) - c_2 q_2\right]$$

$$\text{s.t.}\ \ q_2 \geq 0$$

1. Form the Lagrangian:

$$\mathcal{L}(q_1, q_2) = \beta_1 \left[B(q_1) - (c_1 q_1 + (c_2 - c_1) q_2)\right] + \beta_2 \left[B(q_2) - c_2 q_2\right].$$

2. Apply Lagrange's theorem to right out the first-order necessary conditions:

$$\partial \mathcal{L} / \partial q_1 = \beta_1 [B'(q_1) - c_1] = 0 \qquad (9.13)$$

$$\partial \mathcal{L} / \partial q_2 = -\beta_1 (c_2 - c_1) + \beta_2 [B'(q_2) - c_2] \leq 0 \text{ and } q_2 \geq 0 \text{ with complementary slackness.} \qquad (9.14)$$

From (9.13), we have

$$B'(q_1) = c_1.$$

Note that it is the same as the condition for Type-1 when cost is observable, see (9.12).

From (9.14),

(i) **Case I: $\mathbf{q_2} = \mathbf{0}$.** Then by (9.14), we must have

$$-\beta_1(c_2 - c_1) + \beta_2[B'(0) - c_2] \le 0 \implies B'(0) \le c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1).$$

(ii) **Case II: $\mathbf{q_2} > \mathbf{0}$.** Then by (9.14), we must have

$$-\beta_1(c_2 - c_1) + \beta_2[B'(q_2) - c_2] = 0 \implies B'(q_2) = c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1). \tag{9.15}$$

For $q_2 > 0$, by strict concavity of $B(q_2)$, we need

$$B'(0) > c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1).$$

Therefore, the solution to the relaxed problem is

$$\begin{cases} B'(q_1) = c_1, \; q_2 = 0 & \text{if } B'(0) \le c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1); \\ B'(q_1) = c_1, \; B'(q_2) = c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1) & \text{if } B'(0) > c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1). \end{cases} \tag{9.16}$$

Next, we show that (9.16) also solves the initial problem. We need to show that (IC_2'), i.e., $q_1 \ge q_2$ holds in (9.16).

(i) **Case I: $\mathbf{q_2} = \mathbf{0}$.** $q_1 > 0$ and $q_2 = 0$, thus $q_1 > q_2$.

(ii) **Case II: $\mathbf{q_2} > \mathbf{0}$.** Since $c_1 < c_2 < c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1)$, we have $B'(q_1) < B'(q_2)$. Strict concavity of $B(q)$ implies $q_1 > q_2$.

Therefore, the solution (9.16) is the solution to the simplifed problem and is thus the solution to the government's maximization problem. Besides, the payments R_1 and R_2 are given by (R_1) and (R_2).

Remark. The logic is similar to "(since you are a student in Economic and Management School of Wuhan university), if you are the best student in Wuhan university, then you are the best student in the Economic and Management school of Wuhan university." If we want to find "the best student in the EMS of Wuhan university", we could relax the problem and search for the best student in the whole university. The relaxed problem is easier to solve, since it involves less constraints. However, the solution to the relaxed problem may not be the solution to the initial problem. You must make sure that the conditions left out are indeed satisfied.

There are several points worth noticing.

1. When $B'(0) \leq c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1)$, the high cost supplier does not produce. This is likely to happen when

   a) β_2 is small: the probability that the supplier is a high cost one is low, and

   b) $B'(0)$ is small: the benefit of having a high cost supplier producing is small.

   If this is the case, by placing $q_2 = 0$, the government can effectively eliminate the incentive of a low cost supplier to pretend to be a high cost one.

2. When $q_2 > 0$, by strict concavity of $B(q)$, the optimal q_2, which is given by (9.15), is lower than the optimal q_2 when cost is observable, which is given by (9.12):

   $$B'(q_2) = c_2.$$

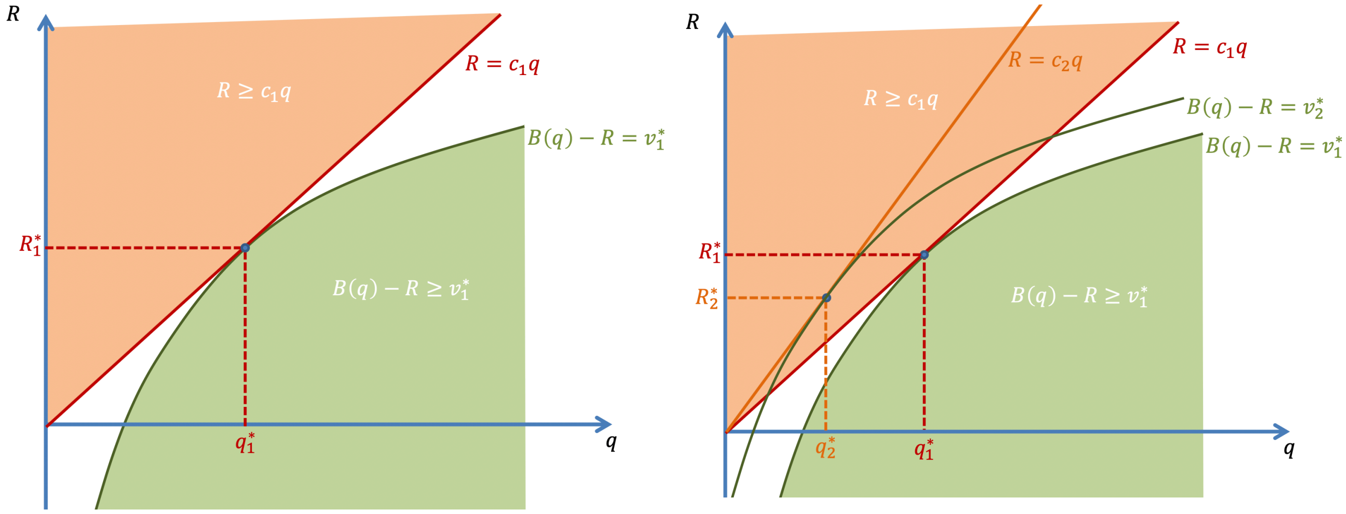
   Lowering the quantities demanded for the high-cost supplier makes it less tempting for the low-cost supplier to declare high cost.

   To elaborate the idea, observe that in equilibrium, the low-cost supplier gets

   $$R_1 - c_1 q_1 \underbrace{=}_{(R_1)} [c_1 q_1 + (c_2 - c_1) q_2] - c_1 q_1 = (c_2 - c_1) q_2.$$

   This gain is referred to as the low-cost supplier's *information rent*. The source of this information rent comes from its ability to mimic high-cost supplier. Lowering q_2 reduces this information rent, and thus reduces its incentive to declare high cost. For the government, lowering q_2 allows it to make less payment R_1 to the low-cost supplier while still keep the low-cost supplier indifferent. The optimal q_2 thus balances the tradeoff between improving efficiency (q_2 closer to the observable cost case) and saving information rent (R_1 smaller).

We could also graphically illustrate the idea. Figure 9.1 below illustrate the cases with observable cost. Figure 9.1a shows the government's maximization problem when the cost is c_1. The orange area is the constraint. The green area is the upper contour set of the objective function. The maximum is attained when the two curves are tangent. In Figure 9.1b, we put the the maximization problem with Type-2 supplier on top of the previous graph. Since $c_1 < c_2$, the line with c_2 is steeper.

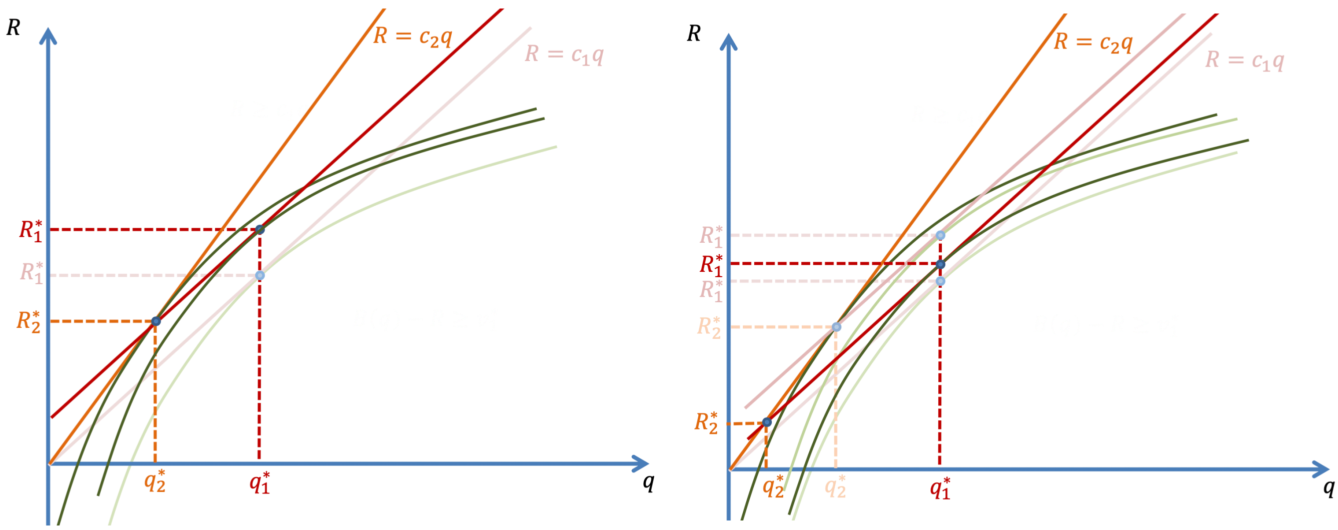


(a) (b)

Figure 9.1: Observable Cost

From Figure 9.1b, the contract for Type-2 supplier (q_2^*, R_2^*) lies inside the orange area. Furthermore, Type-1 supplier prefers this contract to the contract designed for him (q_1^*, R_1^*). Therefore, if the government cannot observe the supplier type, the contracts in Figure 9.1b will not be incentive compatible.

To make the contract incentive compatible when costs are unobservable, one simple way is to keep (q_2^*, R_2^*) unchanged, and move the red line upward so that Type-1 supplier is indifferent between the two contracts. This is shown in Figure 9.2a. Now Type-1 supplier gets positive surplus, which is referred to as *information rent.*



(a) (b)

Figure 9.2: Unobservable Cost

The government can do better than this. It is possible to reduce the information rent left to the low cost supplier. However, there is a price associated with such an act. To keep the contract incentive compatible to the low cost supplier, (q_2^*, R_2^*) must move away from the original optimal contract, that is, the *efficiency* is decreased. The *efficiency* concern and the *rent* extraction is the key trade-off faced by the government. The optimal solution is the one we calculated before.

Appendix A

Solving the maximization problem using the Lagrange's method.

$$\max_{q_1,q_2} \beta_1 \left[B(q_1) - (c_1 q_1 + (c_2 - c_1) q_2)\right] + \beta_2 \left[B(q_2) - c_2 q_2\right]$$

$$\text{s.t. } q_1 \geq q_2 \qquad (IC_2\text{'})$$

$$q_2 \geq 0$$

1. Form the Lagrangian:

$$\mathcal{L}(q_1, q_2, \lambda) = \beta_1 \left[B(q_1) - (c_1 q_1 + (c_2 - c_1) q_2)\right] + \beta_2 \left[B(q_2) - c_2 q_2\right] + \lambda(q_1 - q_2).$$

2. Apply Kuhn-Tucker theorem to write out the first-order necessary conditions:

$$\partial\mathcal{L}/\partial q_1 = \beta_1[B'(q_1) - c_1] + \lambda = 0 \qquad (9.17)$$

$$\partial\mathcal{L}/\partial q_2 = -\beta_1(c_2 - c_1) + \beta_2[B'(q_2) - c_2] - \lambda \leq 0 \text{ and } q_2 \geq 0 \text{ with complementary slackness;} \qquad (9.18)$$

$$\partial\mathcal{L}/\partial\lambda = q_1 - q_2 \geq 0 \text{ and } \lambda \geq 0 \text{ with complementary slackness.} \qquad (9.19)$$

Next, we solve the problem using conditions (9.17), (9.18) and (9.19).

(i) **Case I: $\mathbf{q_2 = 0}$.**

a) **Subcase I: $\mathbf{q_1 = q_2 = 0}$.** Then (9.18) becomes

$$-\beta_1(c_2 - c_1) + \beta_2[B'(0) - c_2] - \lambda \leq 0$$

$$\underbrace{\Longrightarrow}_{(9.17)} -\beta_1(c_2 - c_1) + \beta_2[B'(0) - c_2] + \beta_1[B'(0) - c_1] \leq 0$$

$$\Longrightarrow B'(0) \leq c_2$$

Contradicting with (A).

b) **Subcase II: $\mathbf{q_1 > q_2 = 0}$.** Since $q_1 > q_2$, by (9.19), we have $\lambda = 0$. Plugging into (9.17), we have

$$B'(q_1) = c_1. \qquad (9.20)$$

We also need to ensure that (9.18) holds:

$$-\beta_1(c_2 - c_1) + \beta_2[B'(0) - c_2] \leq 0 \implies B'(0) \leq c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1).$$

(ii) **Case II: $\mathbf{q_2 > 0}$.** (9.18) becomes

$$\begin{aligned}
&-\beta_1(c_2 - c_1) + \beta_2[B'(q_2) - c_2] - \lambda = 0 \\
\underbrace{\implies}_{(9.17)} &-\beta_1(c_2 - c_1) + (1 - \beta_1)[B'(q_2) - c_2] + \beta_1[B'(q_1) - c_1] = 0 \\
\implies &(1 - \beta_1)B'(q_2) + \beta_1 B'(q_1) - c_2 = 0 \qquad (9.18')
\end{aligned}$$

a) **Subcase I: $\mathbf{q_1 = q_2 > 0}$.** (9.18') becomes

$$B'(q_1) = c_2.$$

Plugging into (9.17), we have $\lambda = -\beta_1[c_2 - c_1] < 0$. Contradicting with (9.19).

b) **Subcase II: $\mathbf{q_1 > q_2 > 0}$.** By (9.19), $\lambda = 0$. Then (9.17) becomes

$$\beta_1[B'(q_1) - c_1] = 0 \implies B'(q_1) = c_1.$$

Plugging into (9.18'), we have

$$B'(q_2) = c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1).$$

For $q_2 > 0$, we need

$$B'(0) > c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1).$$

To conclude, the solution is

$$\begin{cases} B'(q_1) = c_1, \; q_2 = 0 & \text{if } B'(0) \leq c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1); \\ B'(q_1) = c_1, \; B'(q_2) = c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1) & \text{if } B'(0) > c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1). \end{cases}$$