Chapter 9. Uncertainty

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Introduction

- This chapter concerns choice under uncertainty.
- It is an important topic in economics, and is of great practical interest.
- In real-life, almost every decision needs to be made under uncertainty.

Introduction

- In this chapter, we will sketch a systematic way of making such decisions.
- In terms of mathematics, there will be nothing new.
- You will see more economic concepts and intuitions.

Introduction

- Let's get familiarized with problems of choice under uncertainty.
- Uncertainty means that you do not anticipate a sure outcome.
- To make our discussion more concrete, we will need to introduce some concepts.

Introduction

Consider the following simple example:

Example 9.1. Suppose that you have access to the following lottery: the lottery pays \$100 with probability 1/4, and pays nothing with the remaining probability. The question is, do you wish to pay \$25 for such a lottery?

Introduction

- This is a problem of choice under uncertainty.
- Outcome is uncertain: you will either get \$100 or \$0.
- There are two important elements in the problem:

(i) Outcomes. In the example, outcomes refer to the state paying *\$100*, and the state paying *\$0*.

(ii) Probabilities associated with outcomes. In the example, 1/4 is the probability associated with the state paying *\$100*, and 3/4 is the probability associated with the state paying *\$0*.

Introduction

- Note that probabilities are objective here, but they could be referred to as subjective probabilities in certain applications.
- Probabilities in a well-defined problem should be
  1. non-negative, and
  2. add up to 1.
- In this example, we could write consumer's utility from lottery could be written as follows: $U(\$100, \$0; 1/4, 3/4)$.

Introduction

- More generally, denote possible outcomes by $Y_1, Y_2, ..., Y_m$.
- Probability associated with outcomes by $p_1, p_2, .., p_m$.
- Utility could be written as

$$U(Y_1, Y_2, ..., Y_m; p_1, p_2, .., p_m).$$

- Next, we will introduce a widely-used method to express utility in a way that more analysis could be performed.

9.A. Expected Utility

- Since probabilities are involved, it is somewhat natural to make use of mathematical expectation, or probability weighted average.
- For instance, in Example 9.1, we could express utility as follows:

$$U(\$100, \$0; 1/4, 3/4) = 1/4U(\$100) + 3/4U(\$0).$$

- This is called the von Neumann-Morgenstern utility function, and is of expected utility form.

Expected Utility

- For a general utility function, expected utility form is expressed as follows:

$$U(Y_1, Y_2, ..., Y_m; p_1, p_2, .., p_m)$$

$$= p_1 U(Y_1) + p_2 U(Y_2) + ... + p_m U(Y_m) = \sum_{i=1}^{m} p_i U(Y_i). \quad (9.1)$$

- This formulation is useful in its simplicity and its ability to capture economically interesting aspects of behavior.
- We will discuss some implications of this representation.

Money Lotteries and Risk-aversion

- Now consider Y_i's as money amounts.
- U is an increasing function.
- Definition of Risk-aversion is intuitive.
- In Example 9.1, the lottery gives in expectation

$$\$100 \times 1/4 + \$0 \times 3/4 = \$25.$$

- A risk-averse individual dislikes risk, and thus prefers sure outcome of \$25 to lottery that gives on average \$25.

Risk-aversion

- In general, for two distinct outcomes Y_1 and Y_2 with (any) positive probability p and $(1-p)$ respectively, a decision maker is risk-averse if

$$U(pY_1 + (1-p)Y_2) > pU(Y_1) + (1-p)U(Y_2).$$

- This is, U is (strictly) concave.

Risk-aversion

- More generally, we could include more than 2 states:
- A decision maker is risk-averse if

$$U(\sum_{i=1}^{m} p_i Y_i) > \sum_{i=1}^{m} p_i U(Y_i). \tag{9.2}$$

- If U is twice differentiable, $U'' < 0$ corresponds to risk-aversion.

Insurance

- Let's now bring back decision variable x, which affect some or all of outcomes and probabilities.
- Suppose $Y_1 < Y_2$, which means first state entails some loss relative to second.
- Y_1 occurs with probability p; Y_2 with probability $(1-p)$.
- A risk-averse decision maker would want to purchase insurance.

Insurance

- Consider an insurance policy that requires
  - an advance payment of x (paid independent of state realization), and
  - gives X if state 1 is realized.
- Suppose the insurance policy is actuarially fair: $pX = x$.
- Actuarially fairness is an outcome of a perfectly competitive insurance industry.
  - Insurance company is risk-free.
  - Zero-profit condition implies $pX = x$.

Insurance

- Decision maker's objective function is

$$\max_{x \geq 0} pU(Y_1 - x + X) + (1-p)U(Y_2 - x)$$

$$\iff \max_{x \geq 0} pU(Y_1 - x + x/p) + (1-p)U(Y_2 - x)$$

- FOC for x gives:

$$pU'(Y_1 - x + x/p)(1/p - 1) - (1-p)U'(Y_2 - x) \leq 0$$

$$\text{and } x \geq 0, \text{ with complementary slackness.} \tag{9.3}$$

Insurance

- When $x = 0$,

$$pU'(Y_1)(1/p - 1) - (1-p)U'(Y_2)$$
$$= (1-p)[U'(Y_1) - U'(Y_2)] \underbrace{>}_{U''<0} 0,$$

  contradicting with (9.3).

- Therefore, we must have $x > 0$ at the optimum.

Insurance

- Since $x > 0$, by (9.3),

$$pU'(Y_1 - x + x/p)(1/p - 1) - (1-p)U'(Y_2 - x) = 0$$

$$\Longrightarrow U'(Y_1 - x + x/p) = U'(Y_2 - x) \tag{9.4}$$

- When $U'' < 0$, objective function is concave in x and FOC is also sufficient.
- FOC (9.4) implies

$$Y_1 - x + x/p = Y_2 - x.$$

Insurance

$Y_1 - x + x/p = Y_2 - x$ is the full-insurance result:

a risk-averse decision maker would buy the actuarially fair insurance to the point where the outcomes in different states are equal.

Care

- Consider again previous problem faced by decision maker, but leave aside insurance for the moment.
- Now suppose that probability of bad outcome (state 1) can be reduced by incurring an expense z in advance.
- Specifically, you could think of it as exercising more care by yourself to reduce probability of being ill.
- In terms of modelling, we make probability p a function of z, and the function is decreasing.

Care

- Objective function is

$$\max_{z \geq 0} \phi(z) \equiv \max_{z} p(z)U(Y_1 - z) + (1 - p(z))U(Y_2 - z)$$

- Then, derivative of $\phi(z)$ gives

$$\phi'(z) = \underbrace{\underbrace{-p'(z)}_{\text{reduction of prob.}} \quad \underbrace{[U(Y_2 - z) - U(Y_1 - z)]}_{\text{utility diff.}}}_{\text{marginal benefit}}$$

$$- \underbrace{\{p(z)U'(Y_1 - z) + (1 - p(z))U'(Y_2 - z)\}}_{\text{marginal cost}}$$

Care

Optimal solution is defined by FOC

$$\phi'(z^*) = 0.$$

Moral Hazard

- Suppose both insurance and care variables are available.
- Interaction between insurance company and decision maker could be formulated as the game in the next page.

Moral Hazard

1. Insurance company sells insurance at constant rate $p(\bar{z})$ per \$1 coverage.
   - If individual purchases x shares, insurance company pays $X = \frac{x}{p(\bar{z})}$ when bad outcome (state 1) occurs.
2. Decision maker chooses how much to purchase x.
3. Decision maker chooses care parameter z.
4. Outcome realized and decision maker gets paid from insurance company if realized state is 1.

Assume insurance policy is actuarially fair (in equilibrium).

Moral Hazard

- Insurance is actuarially fair $\implies p(z^*)X = x$, where z^* is decision maker's actual choice
- Since $X = \frac{x}{p(\bar{z})}$, we have $\bar{z} = z^*$.
- Objective function for decision maker is

$$\max_{x\geq 0, z\geq 0} \phi(x,z) \equiv \max_{x\geq 0, z\geq 0} p(z)U(Y_1 - z - x + x/p(z^*)) + (1-p(z))U(Y_2 - z - x)$$

Moral Hazard

- Partial derivative of $\phi(x, z)$ with respect to x gives

$$\phi_x(x, z) = p(z)U'(Y_1 - z - x + x/p(z^*))(1/p(z^*) - 1) - (1 - p(z))U'(Y_2 - z - x)$$

- Next, we show that optimal $x^* > 0$ must hold.

$$\phi_x(0, z^*) = (1 - p(z^*))[U'(Y_1 - z^*) - U'(Y_2 - z^*)] \underbrace{>}_{U''<0} 0.$$

Moral Hazard

- FOC on x gives

$$\phi_x(x^*, z^*) = 0$$

$$\Longrightarrow Y_1 - z^* - x^* + x^*/p(z^*) = Y_2 - z^* - x^* \qquad (9.5)$$

- Optimal choices of x^* and z^* must satisfy (9.5) above.
- Let $Y_1 - z^* - x^* + x^*/p(z^*) = Y_2 - z^* - x^* = Y_0$.

Moral Hazard

- Partial derivatvie of $\phi_z(x, z)$ with respect to z gives

$$\phi_z(x,z) = \underbrace{-p'(z)\left[U(Y_2 - z - x) - U(Y_1 - z - x + x/p(z^*))\right]}_{\text{marginal benefit}}$$

$$-\underbrace{\left[p(z)U'(Y_1 - z - x + x/p(z^*)) + (1-p(z))U'(Y_2 - z - x)\right]}_{\text{marginal cost}}$$

- Evaluated at optimal level (x^*, z^*), we have

$$\phi_z(x^*, z^*) = -p'(z^*) \cdot 0 - U'(Y_0) = -U'(Y_0) < 0.$$

Moral Hazard

- Optimum of care occurs at the corner $z^* = 0$.
- This is known as "moral hazard": availability of full insurance destroys incentive to exercise costly care.

9.B. One Safe and One Risky Asset

- Next, we will study portfolio choice.
- Since we will be working with multiple states, for convenience, we introduce a continuous representation.
- Index i is replaced by a continuous random variable r with support $[\underline{r}, \overline{r}]$.

One Safe and One Risky Asset

- Expected utility form in (9.1) is modified by replacing probabilities with densities, and sums with integrals:

$$\mathbb{E}[U(Y)] = \int_{\underline{r}}^{\overline{r}} U(Y(r))f(r)\mathrm{d}r.$$

- Interpretation of risk-aversion parallels (9.2): A decision-maker is risk averse if

$$\begin{aligned} & U(\mathbb{E}(Y)) > \mathbb{E}[U(Y)] \\ \iff & U(\int_{\underline{r}}^{\overline{r}} Y(r)f(r)\mathrm{d}r) > \int_{\underline{r}}^{\overline{r}} U(Y(r))f(r)\mathrm{d}r. \end{aligned}$$

One Safe and One Risky Asset

A risk-averse investor has initial wealth W_0, and has two investment options:

(i) A risky asset: investing x gives $x(1+r)$, where r is a random variable with density $f(r)$ and support $[\underline{r}, \overline{r}]$.

Assume

- $\mathbb{E}[r] > 0$: on average return is positive
- $\underline{r} < 0$: does not always generate positive return.

(ii) A safe asset: investing x gives x.

One Safe and One Risky Asset

- Investing $x \in [0, W_0]$ in risky asset and rest in safe asset generates final wealth

$$W = x(1+r) + (W_0 - x) = W_0 + xr.$$

- Investor's objective is to maximize expected final wealth:

$$\max_{x \in [0, W_0]} \mathbb{E}[(U(W))] \equiv \max_x \int_{\underline{r}}^{\overline{r}} U(W_0 + xr) f(r) \mathrm{d}r.$$

- Let $\phi(x) = \mathbb{E}[(U(W))]$.

One Safe and One Risky Asset

- Derivative of $\phi(x)$ gives

$$\phi'(x) = \int_{\underline{r}}^{\overline{r}} rU'(W_0 + xr)f(r)\mathrm{d}r.$$

- Note when $x = 0$:

$$\phi'(0) = U'(W_0)\mathbb{E}[r] > 0.$$

- So, $x = 0$ is not optimal.
- Therefore, risk-averse investor will buy at least some of actuarially good investment.

One Safe and One Risky Asset

- Typically, investor will hold some of each asset.
- FOC is

$$\phi'(x) = \int_{\underline{r}}^{\overline{r}} rU'(W_0 + xr)f(r)\mathrm{d}r = 0. \tag{9.6}$$

- If there is an $x < W_0$ satisfying this, then strict concavity of U guarantees that it is global maximum:

$$\phi''(x) = \int_{\underline{r}}^{\overline{r}} r^2 \underbrace{U''(W_0 + xr)}_{U''(W)<0 \text{ for all } W} f(r)\mathrm{d}r < 0. \tag{9.7}$$

One Safe and One Risky Asset: Comparative Statics

- Next, assuming an interior maximum, we consider comparative statics of x with respect to W_0.
- That is, whether investor would invest more or less in the risky asset when he becomes wealthier.
- Now, we recognize W_0 as a parameter in ϕ, i.e., $\phi(x, W_0)$.

One Safe and One Risky Asset: Comparative Statics

- FOC for an interior solution is

$$\phi_x(x, W_0) = 0.$$

- Total differentiation gives

$$\phi_{xx}(x, W_0)\mathrm{d}x + \phi_{xw}(x, W_0)\mathrm{d}W_0 = 0$$

$$\implies \mathrm{d}x/\mathrm{d}W_0 = -\phi_{xw}(x, W_0)/\phi_{xx}(x, W_0).$$

- By second-order sufficient condition, $\phi_{xx}(x, W_0) < 0$.
- Then, sign of $\mathrm{d}x/\mathrm{d}W_0$ is same as:

$$\phi_{xw}(x, W_0) = \int_{\underline{r}}^{\overline{r}} rU''(W_0 + xr)f(r)\mathrm{d}r.$$

One Safe and One Risky Asset: Comparative Statics

- To gain more insight, we introduce a measure of risk-aversion, called absolute risk-aversion:

$$A(W) = -U''(W)/U'(W). \tag{9.8}$$

- Experimental and empirical evidence is consistent with $A(W)$ being decreasing in W.
- If $A(W)$ is decreasing in W, then we would be able to show $\phi_{xw}(x, W_0) > 0$.

One Safe and One Risky Asset: Comparative Statics

- $\phi_{xw}(x, W_0) > 0$ implies $\mathrm{d}x/\mathrm{d}W_0 > 0$.
- Investor would invest more in risky asset when he becomes wealthier.

9.C. Examples

Example 9.1: Managerial Incentives

- A risk-neutral owner (she) has to hire a risk-neutral manager (he) to run a project.
- If project succeeds, it will produce value V.
- Success probability depends on manager's effort.
- Project succeeds with

$$\begin{cases} \text{probability } p & \text{if manager exerts effort;} \\ \text{probability } q\ (< p) & \text{if no effort.} \end{cases}$$

- Effort cost is e.

Example 9.1: Managerial Incentives

- To make it worthwhile to exert effort, suppose that exerting effort generates higher total surplus:

$$pV - e > qV \implies (p-q)V > e. \tag{9.9}$$

- Assume the manager's outside job pays him w.

Example 9.1: Managerial Incentives

What is optimal compensation scheme when

(i) Owner can observe manager's effort?

(ii) Owner cannot observe manager's effort?

Example 9.1: Solution (Case I: Observable Effort)

- Let payment to manager be W, paid when effort exerted.
- Manager is willing to work for owner and exert effort if

$$W - e \geq w \implies W \geq e + w.$$

- After paying least amount W to manager, owner gets

$$pV - W = pV - e - w.$$

- Owner thus is willing to hire manager if

$$w < pV - e. \tag{9.10}$$

Example 9.1: Solution (Case I: Observable Effort)

- (9.10) is an assumption that we would make throughout analysis, since otherwise, manager would not be hired.
- Under assumption, owner could offer $w + e$ to manager, and demand effort in return.
- Owner would get $pV - e - w$ and manager gets w, same as what he would get from outside job.

Example 9.1: Solution (Case II: Unobservable Effort)

In this case, compensating effort directly would not work.

- Suppose that compensation is still based on (now unoberservable) effort, then manager could lie about effort: manager could promise to exert effort, but shirk instead.
- Because of
  1. unobservability of effort and
  2. probalistic nature of the outcome,

  owner would not catch such a lie.

Example 9.1: Solution (Case II: Unobservable Effort)

- Therefore, the best thing owner could do is to base his payment scheme on the thing that he could observe, i.e., outcome.
- Suppose that owner pays manager
  - x if the project succeeds, and
  - y if it fails.

Example 9.1: Solution (Case II: Unobservable Effort)

Two constraints need to be satisfied:

(i) Given such a payment scheme, manager would exert effort

if $$px + (1-p)y - e \geq qx + (1-q)y$$

$$\Longrightarrow (p-q)(x-y) \geq e. \qquad \text{(IC)}$$

This is called incentive compatibility constraint.

(ii) Manager will agree to work if

$$px + (1-p)y - e \geq w \implies y + p(x-y) \geq w + e. \qquad \text{(IR)}$$

This is called participation constraint or individual rationality constraint.

Example 9.1: Solution (Case II: Unobservable Effort)

- Thus, owner's problem is to maximize her profit subject to constraints (IC) and (IR).

$$\max_{x,y} pV - [px + (1-p)y] \equiv \max_{x,y} pV - y - p(x-y)$$

$$\text{s.t. } (p-q)(x-y) \geq e; \qquad \text{(IC)}$$

$$y + p(x-y) \geq w + e. \qquad \text{(IR)}$$

- (IR) must be binding.

Example 9.1: Solution (Case II: Unobservable Effort)

- From IC and binding IR, we know

$$y^* \le w - eq/(p-q) \text{ and } x^* \ge w + e(1-q)/(p-q).$$

- One interpretation is that manager's compensation consists of
  - basic salary w,
  - reward for success and penalty for failure.
- By binding IR, owner's expected profit is

$$\pi = pV - y^* - p(x^* - y^*) = pV - w - e,$$

same as when she could observe manager's effort.

Example 9.1: Solution (Case II: Unobservable Effort)

- However, one potential problem here is that

$$y^*_{\max} = w - eq/(p - q)$$

  is not guaranteed to be positive.

- $y^*_{\max} < 0$ means that payment scheme would involve a fine under failure, which is always not feasible.

Example 9.1: Solution (Case II: Unobservable Effort)

- Suppose $y^*_{\max} = w - eq/(p-q) < 0$ and $y \geq 0$ is required.
- Solution is to go as far as possible, i.e., $y = 0$.
- (IC) and (IR) becomes

$$(p-q)(x-0) \geq e \implies x \geq \frac{e}{p-q}; \qquad \text{(IC')}$$

$$0 + p(x-0) \geq w + e \implies x \geq \frac{w+e}{p}. \qquad \text{(IR')}$$

- Problem becomes:

$$\max_x pV - px$$

$$\text{s.t. } (IC') \ \& \ (IR')$$

- Owner wants x to be as small as possible.

Example 9.1: Solution (Case II: Unobservable Effort)

- From $y^*_{\max} = w - eq/(p-q) < 0$, we have

$$(w+e)/p < e/(p-q). \tag{9.11}$$

- Minimum x is $x^{**} = e/(p-q)$.
- Profit becomes $\pi = pV - px^{**} = pV - pe/(p-q)$.
- By (9.11), this profit level is lower compared to previous cases where $\pi = pV - w - e$.
- By (9.9), this profit level is still positive.

Example 9.2: Cost-Plus Contracts

- This example is motivated by cost-plus contract.
- Government expenditures are often made on such a cost-plus basis, that is, government reimburses supplier's cost plus a normal profit.
- In this example, we are concerned with appropriated amount of reimbursement when government does not observe supplier's cost.

Example 9.2: Cost-Plus Contracts

- Suppose true average cost of production can take just two values: c_1 and c_2 (normal profit included), with $c_1 < c_2$.
- We call supplier with cost c_i Type-i supplier.
- Supplier is privately informed of its own type.
- Before contracting, government's estimate of probability of supplier being Type-1 is β_1, Type-2 is $\beta_2 = 1 - \beta_1$.
- Problem here: low cost supplier would pretend to be of high cost and get more reimbursement from government.

Example 9.2: Cost-Plus Contracts

- Government would offer contracts: if supplier claims to have cost c_i for $i = 1, 2$, government
  - purchases q_i units and
  - pays R_i.
- In game theory, use of different contracts to separate supplier types is called "screening".

Example 9.2: Cost-Plus Contracts

- Governments gets benefit $B(q)$ from quantity q.
- $B(q)$ is strictly increasing, strictly concave in q, and

$$B'(0) > c_2 \tag{A}$$

so that government would demand positive quantities from either type if cost can be observed.

What is the optimal menu of contracts (q_1, R_1) and (q_2, R_2) when cost is unobservable?

Example 9.2: Solution (Observable Cost)

- Government problem facing a supplier with Type-i is:

$$\max_{q_i, R_i} B(q_i) - R_i$$

$$\text{s.t. } R_i - c_i q_i \geq 0. \tag{IR}$$

$$q_i \geq 0, R_i \geq 0.$$

- Before solving the problem, we make two observations:

1. (IR) must be binding.

2. (IR) and $q_i \geq 0$ implies $R_i \geq 0$.

Example 9.2: Solution (Observable Cost)

- Government's problem is reduced to

$$\max_{q_i} B(q_i) - c_i q_i$$

$$\text{s.t. } q_i \geq 0.$$

- Optimal q_i is given by

$$B'(q_i) = c_i. \tag{9.12}$$

- Optimal R_i is given by binding (IR)

$$R_i - c_i q_i = 0. \tag{IR}$$

Example 9.2: Solution (Unobservable Cost)

When cost is unobservable, government would offer two contracts and let supplier choose.

Example 9.2: Solution (Unobservable Cost)

- To make supplier willing to choose contract designed for his type:

$$R_1 - c_1 q_1 \geq R_2 - c_1 q_2; \qquad (IC_1)$$

$$R_2 - c_2 q_2 \geq R_1 - c_2 q_1. \qquad (IC_2)$$

- These are incentive compatibility constraints.

Example 9.2: Solution (Unobservable Cost)

- To ensure that supplier wants to participate:

$$R_1 - c_1 q_1 \geq 0; \qquad (IR_1)$$

$$R_2 - c_2 q_2 \geq 0. \qquad (IR_2)$$

- These are participation constraints.

Example 9.2: Solution (Unobservable Cost)

- We also need to ensure $q_i \geq 0$ and $R_i \geq 0$.
- Government's problem is

$$\max_{q_1, q_2, R_1, R_2} \beta_1 [B(q_1) - R_1] + \beta_2 [B(q_2) - R_2]$$

$$\text{s.t. } (IC_1), (IC_2), (IR_1), (IR_2)$$

$$q_1 \geq 0, q_2 \geq 0, R_1 \geq 0, R_2 \geq 0.$$

Example 9.2: Solution (Unobservable Cost)

- It is a maximization problem with 4 inequality constraints and 4 non-zero variables.
- These inequality pairs permit $2^8 = 256$ patterns.
- We will make some initial analysis to simplify problem.

Example 9.2: Solution (Unobservable Cost)

Lemma 1. *$R_i \geq 0$ is implied by (IR_1), (IR_2) and $q_i \geq 0$.*

- We could safely ignore non-negativity constraints on R_i.

Example 9.2: Solution (Unobservable Cost)

Lemma 2. *(IR_1) is implied by (IC_1), (IR_2) and $q_2 \geq 0$.*

- We could safely ignore (IR_1).

Example 9.2: Solution (Unobservable Cost)

From Lemmas 1 and 2, government's problem becomes:

$$\max_{q_1,q_2,R_1,R_2} \beta_1 [B(q_1) - R_1] + \beta_2 [B(q_2) - R_2]$$

$$\text{s.t. } R_1 - c_1 q_1 \geq R_2 - c_1 q_2; \qquad (IC_1)$$

$$R_2 - c_2 q_2 \geq R_1 - c_2 q_1; \qquad (IC_2)$$

$$R_2 - c_2 q_2 \geq 0; \qquad (IR_2)$$

$$q_1 \geq 0, q_2 \geq 0.$$

Example 9.2: Solution (Unobservable Cost)

Lemma 3. *(IR_2) must be binding in optimal scheme, i.e.,*

$R_2 - c_2 q_2 = 0.$

Example 9.2: Solution (Unobservable Cost)

Lemma 4. *(IC_1) must be binding in the optimal scheme, i.e., $R_1 - c_1 q_1 = R_2 - c_1 q_2$.*

Example 9.2: Solution (Unobservable Cost)

- By Lemmas 3 and 4, we have

$$R_2 = c_2 q_2 \tag{R_2}$$

$$R_1 = c_1 q_1 + (c_2 - c_1) q_2 \tag{R_1}$$

- Plugging (R_2) and (R_1) into (IC_2), we have

$$q_1 \geq q_2 \tag{IC_2'}$$

Example 9.2: Solution (Unobservable Cost)

- Maximization problem is simplfied as follows:

$$\max_{q_1,q_2} \beta_1 \left[B(q_1) - (c_1 q_1 + (c_2 - c_1) q_2)\right] + \beta_2 \left[B(q_2) - c_2 q_2\right]$$

$$\text{s.t. } q_1 \geq q_2 \qquad (IC_2\text{'})$$

$$q_2 \geq 0$$

- We can solve this problem in usual way.
- See Appendix A.

Example 9.2: Solution (Unobservable Cost)

- Here, we introduce another way of solving this problem:
  1. solve relaxed problem with no (IC_2'), and
  2. show that solution to relaxed problem is also solution to initial problem.
- Relaxed problem is as follows:

$$\max_{q_1,q_2} \beta_1 \left[B(q_1) - (c_1 q_1 + (c_2 - c_1) q_2)\right] + \beta_2 \left[B(q_2) - c_2 q_2\right]$$

$$\text{s.t. } q_2 \geq 0$$

Example 9.2: Solution (Unobservable Cost)

Solution to relaxed problem is

$$\begin{cases} B'(q_1) = c_1,\ q_2 = 0 \\ \qquad\qquad \text{if } B'(0) \le c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1); \\ B'(q_1) = c_1,\ B'(q_2) = c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1) \\ \qquad\qquad \text{if } B'(0) > c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1). \end{cases} \tag{9.16}$$

Example 9.2: Solution (Unobservable Cost)

- Next, we show that (9.16) also solves initial problem.
- We need to show: (IC_2') holds in (9.16).

Example 9.2: Solution (Unobservable Cost)

- The logic is similar to "(since you are a student in Economic and Management School of Wuhan university), if you are the best student in Wuhan university, then you are the best student in EMS of Wuhan university."
- If we want to find "the best student in EMS of Wuhan university", we could relax the problem and search for the best student in whole university.

Example 9.2: Solution (Unobservable Cost)

- Relaxed problem is easier to solve since it involves less constraints.
- However, solution to relaxed problem may not be solution to initial problem.
- You must make sure that conditions left out are indeed satisfied.

Example 9.2: Solution (Unobservable Cost)

There are two points worth noticing.

1. When $B'(0) \leq c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1)$, high cost supplier does not produce. This is likely to happen when
   a) β_2 is small: probability of high cost is low,
   b) $B'(0)$ is small: benefit of having a high cost supplier producing a little is small.

- By placing $q_2 = 0$, government can effectively eliminate incentive of low cost supplier to pretend to be high cost.

Example 9.2: Solution (Unobservable Cost)

2. When $q_2 > 0$, optimal q_2:

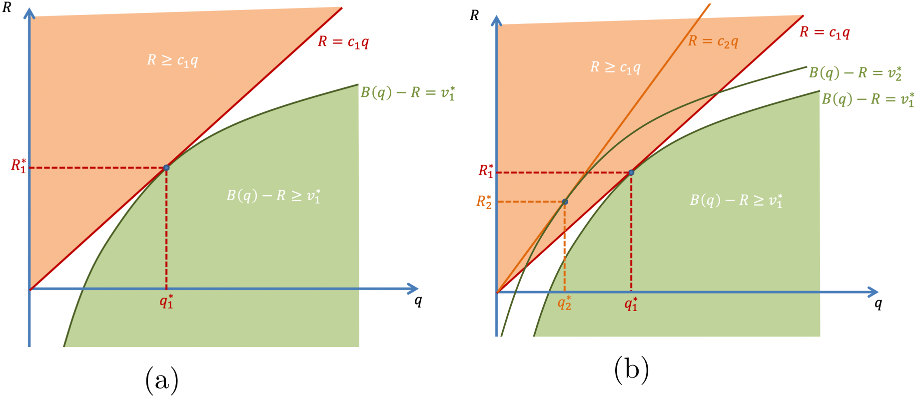
$$B'(q_2) = c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1).$$

   is lower than optimal q_2 when cost is observable:

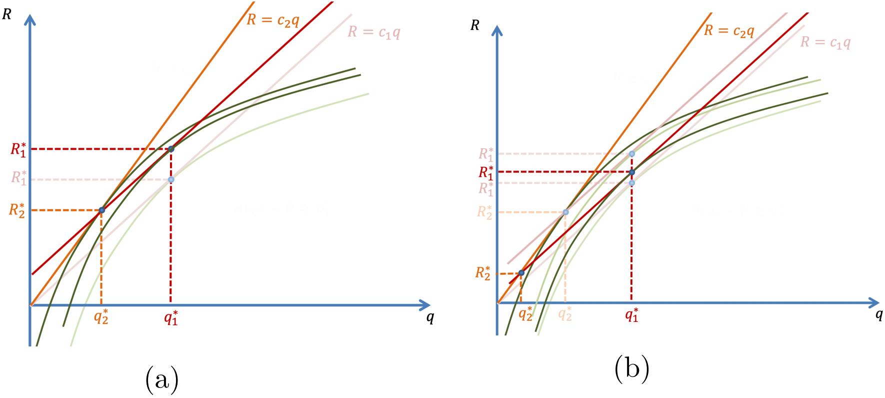
$$B'(q_2) = c_2.$$

- Lowering quantities demanded for high-cost supplier makes it less tempting for low-cost supplier to declare high cost.

Example 9.2: Graphs (Observable Cost)



Example 9.2: Graphs (Unobservable Cost)



Efficiency concern and rent extraction is key trade-off faced by government.